ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$ then:

$$\sqrt{3a^2+2ab+3b^2}+\sqrt{3b^2+2bc+3c^2}+\sqrt{3c^2+2ca+3a^2}\ge 6\sqrt{2}$$

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$$\sqrt{a}+\sqrt{b}+\sqrt{c}=3\ \ or\ \sqrt{3(a+b+c)}\overset{CBS}{\ge}\ 3$$

$$3(a+b+c)\ge 9\ or\ \ a+b+c\ge 3\ \ (1)$$

$Lemma$:

$$\forall x,y>0\ ,\sqrt{3x^2+2xy+3y^2}\ge\sqrt{2}(x+y)$$

$Proof$:

$$We\ need\ to\ show:$$

$$\sqrt{3x^2+2xy+3y^2}\ge\sqrt{2}(x+y)$$

$$3x^2+2xy+3y^2\ \overset{squaring}{\ge}\ 2(x^2+2xy+y^2)$$

$$x^2-2xy+y^2\ge 0\ or\ (x-y)^2\ge 0\ true$$

$$\sqrt{3a^2+2ab+3b^2}+\sqrt{3b^2+2bc+3c^2}+\sqrt{3c^2+2ca+3a^2}=$$

$$=\sum\sqrt{3a^2+2ab+3b^2}\ \overset{lemma}{\ge}\ \sum\sqrt{2}(a+b)=2\sqrt{2}(a+b+c)\overset{(1)}{\ge}\ 6\sqrt{2}$$

Equality holds for a=b=c=1.