ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0,\ a+b+c=3$ then:

$$\frac{x^2}{\sqrt{15x^2+26xy+8y^2}}+\frac{y^2}{\sqrt{15y^2+26yz+8z^2}}+\frac{z^2}{\sqrt{15z^2+26zx+8x^2}}\ge\frac{3}{7}$$

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Lemma:

$$\forall\ a,b>0\ ,\sqrt{15a^2+26ab+8b^2}\le 4a+3b$$

Proof:

$$We\ need\ to\ show:$$

$$\sqrt{15a^2+26ab+8b^2}\le 4a+3b$$

$$or,15a^2+26ab+8b^2\overset{squaring}{\le}16a^2+24ab+9b^2$$

$$or,a^2-2ab+b^2\ge 0\ or\ (a-b)^2\ge 0\ true.$$

$$\frac{x^2}{\sqrt{15x^2+26xy+8y^2}}+\frac{y^2}{\sqrt{15y^2+26yz+8z^2}}+\frac{z^2}{\sqrt{15z^2+26zx+8x^2}}=$$

$$=\sum\frac{x^2}{\sqrt{15x^2+26xy+8y^2}}\overset{Lemma}{\ge}\sum\frac{x^2}{4x+3y}\overset{Bergstrom}{\ge}$$

$$\ge\frac{(x+y+z)^2}{4(x+y+z)+3(x+y+z)}=\frac{x+y+z}{7}\overset{x+y+z=3}{=}\frac{3}{7}$$

$$Equality\ holds\ for\ x=y=z=1.$$