ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0,\ \sqrt{a+1}+\sqrt{b+1}+\sqrt{c+1}=6$ then:

$$\sqrt{a^2+ab+b^2}+\sqrt{b^2+bc+b^2}+\sqrt{c^2+ca+a^2}\ge 9\sqrt{3}$$

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$$\sqrt{a+1}+\sqrt{b+1}+\sqrt{c+1}=6\ or$$
$$\sqrt{3(a+1+b+1+c+1)}\overset{CBS}{\ge}6\ or\ 3(a+b+c+3)\overset{squaring}{\ge}36$$
$$or\ a+b+c\ge 9\ (1)$$

Lemma:

$$\forall x,y>0\ ,\sqrt{x^2+xy+y^2}\ge\frac{\sqrt{3}}{2}(x+y)$$

Proof:

$$We\ need\ to\ show:$$
$$\sqrt{x^2+xy+y^2}\ge\frac{\sqrt{3}}{2}(x+y)\ \ or\ x^2+xy+y^2\overset{squring}{\ge}\frac{3}{4}(x+y)^2$$
$$4(x^2+y^2+xy)\ge 3(x^2+y^2+2xy)\ or$$
$$x^2-2xy+y^2\ge 0\ or(x-y)^2\ge 0\ true$$

$$\sqrt{a^2+ab+b^2}+\sqrt{b^2+bc+b^2}+\sqrt{c^2+ca+a^2}=$$
$$=\sum\sqrt{a^2+ab+b^2}\overset{lemma}{\ge}\sum\frac{\sqrt{3}}{2}(a+b)=\sqrt{3}(a+b+c)\overset{(1)}{\ge}9\sqrt{3}.$$
$$Equality\ holds\ for\ a=b=c=3$$