ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ then:

$$\sqrt{5x^2+14xy+5y^2}+\sqrt{5y^2+14zy+5z^2}+\sqrt{5z^2+14xz+5x^2}\le 2\sqrt{6}(x+y+z)$$

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$\boldsymbol{Lemma}$:

$$\forall a, b > 0,\ \sqrt{5a^2+14ab+5b^2}\le\sqrt{6}(a+b)$$

$\boldsymbol{Proof}$:

$$\boldsymbol{We\ need\ to\ show}:$$

$$\sqrt{5a^2+14ab+5b^2}\le\sqrt{6}(a+b)$$

$$5a^2+14ab+5b^2 \overset{squaring}{\le} 6(a+b)^2$$

$$5a^2+14ab+5b^2\le 6(a^2+2ab+b^2)$$

$$a^2-2ab+b^2\ge 0\ or\ (a-b)^2\ge 0\ true$$

$$\sqrt{5x^2+14xy+5y^2}+\sqrt{5y^2+14zy+5z^2}+\sqrt{5z^2+14xz+5x^2}=$$

$$=\sum\sqrt{5x^2+14xy+5y^2}\overset{lemma}{\le}\sum\sqrt{6}(x+y)=2\sqrt{6}(x+y+z)$$

Equality holds for x=y=z.