ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ then:

$$\sqrt{2x^2+xy+2y^2}+\sqrt{2y^2+zy+2z^2}+\sqrt{2z^2+xz+2x^2}\ge\sqrt{5}(x+y+z)$$

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$\boldsymbol{Lemma}$:

$$\forall a,b>0\ ,\sqrt{2a^2+ab+2b^2}\ge\frac{\sqrt{5}}{2}(a+b)$$

$\boldsymbol{Proof}$:

$$\textit{We need to show}:$$

$$\sqrt{2a^2+ab+2b^2}\ge\frac{\sqrt{5}}{2}(a+b)$$

$$4(2a^2+ab+2b^2)\overset{squaring}{\ge}5(a+b)^2$$

$$8a^2+4ab+8b^2\ge5(a^2+2ab+b^2)$$

$$3(a^2-2ab+b^2)\ge0\ or\ 3(a-b)^2\ge0\ \ true$$

$$\sqrt{2x^2+xy+2y^2}+\sqrt{2y^2+zy+2z^2}+\sqrt{2z^2+xz+2x^2}=$$

$$=\sum\sqrt{2x^2+xy+2y^2}\overset{Lemma}{\ge}\sum\frac{\sqrt{5}}{2}(x+y)\ =\sqrt{5}(x+y+z)$$

$$\textit{Equality holds for } x=y=z.$$