ROMANIAN MATHEMATICAL MAGAZINE

If $x\sqrt{y^2+2}+y\sqrt{x^2+2}=4\sqrt{6}$ then prove that :

$$\sqrt{x^2+2}+\sqrt{y^2+2}-x-y\geq 2\sqrt{6}-4$$

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$$\left(x.\sqrt{y^2+2}\right)^2=\left(4\sqrt{6}-y.\sqrt{x^2+2}\right)^2$$

$$\Rightarrow x^2(y^2+2)=96+x^2y^2+2y^2-8y.\sqrt{6x^2+12}$$

$$\Rightarrow y^2-4y.\sqrt{6x^2+12}+48-x^2=0$$

$$\Rightarrow y=\frac{4.\sqrt{6x^2+12}\pm\sqrt{96x^2+192-192+4x^2}}{2}$$

$$\Rightarrow y=2.\sqrt{6x^2+12}+5x \text{ or } y=2.\sqrt{6x^2+12}-5x$$

$\boxed{\text{Case 1}}$ If $y=2.\sqrt{6x^2+12}+5x$, then :

$$x.\sqrt{y^2+2}+\left(2.\sqrt{6x^2+12}+5x\right).\sqrt{x^2+2}=4\sqrt{6}$$

$$\Rightarrow x.\sqrt{y^2+2}+2\sqrt{6}.(x^2+2)+5x.\sqrt{x^2+2}=4\sqrt{6}$$

$$\Rightarrow x\left(\sqrt{y^2+2}+2\sqrt{6}.x+5.\sqrt{x^2+2}\right)\overset{(*)}{=}0 \text{ and when } x\geq 0,$$

$$\sqrt{y^2+2}+2\sqrt{6}.x+5.\sqrt{x^2+2} \text{ is trivially} >0 \text{ and when } x<0,$$

$$\sqrt{y^2+2}+2\sqrt{6}.x+5.\sqrt{x^2+2}=\sqrt{y^2+2}+\frac{25x^2+50-24x^2}{5.\sqrt{x^2+2}-2\sqrt{6}.x}$$

$$=\sqrt{y^2+2}+\frac{x^2+50}{5.\sqrt{x^2+2}-2\sqrt{6}.x}>0 \text{ and so}, \sqrt{y^2+2}+2\sqrt{6}.x+5.\sqrt{x^2+2}>0$$

$$\forall\, x,y\in\mathbb{R} \text{ and so}, (*)\Rightarrow x=0 \text{ and then : } y=4\sqrt{3}$$

$$\Rightarrow \sqrt{x^2+2}+\sqrt{y^2+2}-x-y-2\sqrt{6}+4=\sqrt{2}+\sqrt{50}-4\sqrt{3}-2\sqrt{6}+4$$

$$=2\left(3\sqrt{2}-2\sqrt{3}-\sqrt{6}+2\right)>0\Rightarrow \sqrt{x^2+2}+\sqrt{y^2+2}-x-y>2\sqrt{6}-4$$

$\boxed{\text{Case 2}}$ If $y\overset{(\bullet)}{=}2.\sqrt{6x^2+12}-5x$, then : $x+y=2.\sqrt{6x^2+12}-4x=t\,(say)$

$$\Rightarrow 24x^2+48=16x^2+t^2+8xt\Rightarrow 8x^2-8t.x+48-t^2=0 \text{ and } \because x\in\mathbb{R},$$

$$\text{discriminant}\geq 0\Rightarrow 64t^2-32(48-t^2)\geq 0\Rightarrow t^2\geq 16$$

$$\Rightarrow \boxed{x+y\geq 4 \text{ or } x+y\leq -4} \text{ and if } t=x+y\leq -4, \text{ then : } 2.\sqrt{6x^2+12}-4x=t$$

$$\leq -4 < 0 \Rightarrow \sqrt{6x^2+12} < 2x \leq 2|x| \Rightarrow 6x^2+12 < 4x^2 \Rightarrow 2x^2+12 < 0$$

$$\to \textbf{impossible} \therefore x+y \geq 4 \to (1) \text{ and also}, y = 2.\sqrt{6x^2+12} - 5x$$

$$\Rightarrow (y+5x)^2 = 4(6x^2+12) \Rightarrow x^2 + 10y.x + y^2 - 48 = 0$$

$$\Rightarrow \boxed{x = -5y \pm 2.\sqrt{6y^2+12}} \text{ and if } x = -5y - 2.\sqrt{6y^2+12}, \text{ then :}$$

$$x+y = -4y - 2.\sqrt{6y^2+12} \text{ and we shall show that : } -4y - 2.\sqrt{6y^2+12} \overset{?}{\leq} -4$$

$$\Leftrightarrow 2(1-y) \overset{?}{\leq} \sqrt{6y^2+12} \text{ and it's trivially true if : } 1-y \leq 0 \text{ and for } 1-y > 0,$$

$$2(1-y) \overset{?}{\leq} \sqrt{6y^2+12} \Leftrightarrow 6y^2+12 \overset{?}{\geq} 4+4y^2-8y \Leftrightarrow 2(y+2)^2 \overset{?}{\geq} 0 \to \text{true}$$

$$\therefore x+y = -4y - 2.\sqrt{6y^2+12} \leq -4; \text{ but } (1) \Rightarrow x+y \geq 4$$

$$\therefore x+y \neq -4y - 2.\sqrt{6y^2+12} \Rightarrow \boxed{x \neq -5y - 2.\sqrt{6y^2+12}}$$

$$\therefore x \overset{(\bullet\bullet)}{=} 2.\sqrt{6y^2+12} - 5y \text{ and so, } (\bullet) \text{ and } (\bullet\bullet) \Rightarrow \sqrt{6x^2+12} + \sqrt{6y^2+12}$$

$$= \frac{y+5x}{2} + \frac{x+5y}{2} = 3(x+y) \Rightarrow \sqrt{6x^2+12} + \sqrt{6y^2+12} - \sqrt{6}.(x+y)$$

$$= (3-\sqrt{6})(x+y) \overset{\text{via } (1)}{\geq} 4(3-\sqrt{6}) = \sqrt{6}.(2\sqrt{6}-4)$$

$$\Rightarrow \sqrt{x^2+2} + \sqrt{y^2+2} - x - y \geq 2\sqrt{6} - 4 \text{ and combining both cases,}$$

$$x.\sqrt{y^2+2} + y.\sqrt{x^2+2} = 4\sqrt{6} \Rightarrow \sqrt{x^2+2} + \sqrt{y^2+2} - x - y \geq 2\sqrt{6} - 4,$$

$$'' ='' \text{ iff } x = y = 2 \text{ (QED)}$$