ROMANIAN MATHEMATICAL MAGAZINE

If $x\sqrt{y^2+1}+y\sqrt{x^2+1}=2\sqrt{2}$ then prove that :

$$\sqrt{x^2+1}+\sqrt{y^2+1}-x-y\geq 2\sqrt{2}-2$$

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$$\left(x.\sqrt{y^2+1}\right)^2=\left(2\sqrt{2}-y.\sqrt{x^2+1}\right)^2$$

$$\Rightarrow x^2(y^2+1)=8+x^2y^2+y^2-4y.\sqrt{2x^2+2}\Rightarrow y^2-4y.\sqrt{2x^2+2}+8-x^2=0$$

$$\Rightarrow y=\frac{4.\sqrt{2x^2+2}\pm\sqrt{32x^2+32-32+4x^2}}{2}$$

$$\Rightarrow y=2.\sqrt{2x^2+2}+3x \text{ or } y=2.\sqrt{2x^2+2}-3x$$

$\boxed{\text{Case 1}}$ If $y=2.\sqrt{2x^2+2}+3x$, then : $x.\sqrt{y^2+1}+\left(2.\sqrt{2x^2+2}+3x\right).\sqrt{x^2+1}$

$$=2\sqrt{2}\Rightarrow x.\sqrt{y^2+1}+2\sqrt{2}.(x^2+1)+3x.\sqrt{x^2+1}=2\sqrt{2}$$

$$\Rightarrow x\left(\sqrt{y^2+1}+2\sqrt{2}.x+3.\sqrt{x^2+1}\right)\overset{(*)}{=}0 \text{ and when } x\geq 0,$$

$$\sqrt{y^2+1}+2\sqrt{2}.x+3.\sqrt{x^2+1} \text{ is trivially} >0 \text{ and when } x<0,$$

$$\sqrt{y^2+1}+2\sqrt{2}.x+3.\sqrt{x^2+1}=\sqrt{y^2+1}+\frac{9x^2+9-8x^2}{3.\sqrt{x^2+1}-2\sqrt{2}.x}$$

$$=\sqrt{y^2+1}+\frac{x^2+9}{3.\sqrt{x^2+1}-2\sqrt{2}.x}>0 \text{ and so, } \sqrt{y^2+1}+2\sqrt{2}.x+3.\sqrt{x^2+1}>0$$

$$\forall\, x,y\in\mathbb{R} \text{ and so, } (*)\Rightarrow x=0 \text{ and then : } y=2\sqrt{2}$$

$$\Rightarrow\sqrt{x^2+1}+\sqrt{y^2+1}-x-y-2\sqrt{2}+2=1+3-2\sqrt{2}-2\sqrt{2}+2$$

$$=2\left(3-2\sqrt{2}\right)>0\Rightarrow\sqrt{x^2+1}+\sqrt{y^2+1}-x-y>2\sqrt{2}-2$$

$\boxed{\text{Case 2}}$ If $y\overset{(\bullet)}{=}2.\sqrt{2x^2+2}-3x$, then : $x+y=2.\sqrt{2x^2+2}-2x=t\,(\text{say})$
$\Rightarrow 8x^2+8=4x^2+t^2+4xt\Rightarrow 4x^2-4t.x+8-t^2=0$ and $\because x\in\mathbb{R}$,
discriminant $\geq 0\Rightarrow 16t^2-16(8-t^2)\geq 0\Rightarrow t^2\geq 4$
$\Rightarrow\boxed{x+y\geq 2 \text{ or } x+y\leq -2}$ and if $t=x+y\leq -2$, then : $2.\sqrt{2x^2+2}-2x=t\leq$
$-2<0\Rightarrow\sqrt{2x^2+2}<x\leq|x|\Rightarrow 2x^2+2<x^2\Rightarrow x^2+2<0\rightarrow$ impossible
$\therefore x+y\geq 2\rightarrow$ 1 and also, $y=2.\sqrt{2x^2+2}-3x\Rightarrow(y+3x)^2=4(2x^2+2)$
$\Rightarrow x^2+6y.x+y^2-8=0\Rightarrow\boxed{x=-3y\pm 2.\sqrt{2y^2+2}}$ and if $x=$

$-3y-2.\sqrt{2y^2+2}$, then : $x+y=-2y-2.\sqrt{2y^2+2}$ and we shall show that :

$$-2y-2.\sqrt{2y^2+2}\overset{?}{\leq}-2 \Leftrightarrow 1-y\overset{?}{\leq}\sqrt{2y^2+2} \text{ and it's trivially true if :}$$

$$1-y\leq 0 \text{ and for } 1-y>0, \text{ then : } 1-y\overset{?}{\leq}\sqrt{2y^2+2} \Leftrightarrow 2y^2+2\overset{?}{\geq}1+y^2-2y$$

$$\Leftrightarrow (y+1)^2\overset{?}{\geq}0 \to \text{true} \therefore x+y=-2y-2.\sqrt{2y^2+2}\leq -2; \text{but } (1) \Rightarrow x+y\geq 2$$

$$\therefore x+y\neq -2y-2.\sqrt{2y^2+2} \Rightarrow \boxed{x\neq -3y-2.\sqrt{2y^2+2}}$$

$$\therefore x\overset{(\bullet\bullet)}{=}2.\sqrt{2y^2+2}-3y \text{ and so, } (\bullet) \text{ and } (\bullet\bullet) \Rightarrow \sqrt{2x^2+2}+\sqrt{2y^2+2}$$

$$=\frac{y+3x}{2}+\frac{x+3y}{2}=2(x+y) \Rightarrow \sqrt{2x^2+2}+\sqrt{2y^2+2}-\sqrt{2}.(x+y)$$

$$=\left(2-\sqrt{2}\right)(x+y)\overset{\text{via } (1)}{\geq}2\left(2-\sqrt{2}\right)=\sqrt{2}.\left(2\sqrt{2}-2\right)$$

$$\Rightarrow \sqrt{x^2+1}+\sqrt{y^2+1}-x-y\geq 2\sqrt{2}-2 \text{ and combining both cases,}$$

$$x.\sqrt{y^2+1}+y.\sqrt{x^2+1}=2\sqrt{2} \Rightarrow \sqrt{x^2+1}+\sqrt{y^2+1}-x-y\geq 2\sqrt{2}-2,$$

$''=''$ iff $x=y=1$ (QED)