ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ then prove that :

$$\frac{a^4}{a^3+ab^2+b^3}+\frac{b^4}{b^3+bc^2+c^3}+\frac{c^4}{c^3+ca^2+a^3}\ge\frac{a+b+c}{3}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS}=\sum_{\text{cyc}}\frac{b\left(b^3+bc^2+c^3-c^2(b+c)\right)}{b^3+bc^2+c^3}=\sum_{\text{cyc}}a-\sum_{\text{cyc}}\frac{bc^2(b+c)}{b^3+bc^2+c^3}\ge$$

$$\sum_{\text{cyc}}a-\sum_{\text{cyc}}\frac{bc^2(b+c)}{bc(b+c)+bc^2}=\sum_{\text{cyc}}a-\sum_{\text{cyc}}\frac{c(b+2c-c)}{b+2c}$$

$$=\sum_{\text{cyc}}a-\sum_{\text{cyc}}a+\frac{c^2}{b+2c}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{\text{cyc}}a\right)^2}{\sum_{\text{cyc}}a+2\sum_{\text{cyc}}a}=\frac{a+b+c}{3}$$

$$\therefore\frac{a^4}{a^3+ab^2+b^3}+\frac{b^4}{b^3+bc^2+c^3}+\frac{c^4}{c^3+ca^2+a^3}\ge\frac{a+b+c}{3}$$

$$\forall a,b,c>0,''='' \text{ iff } a=b=c\ (\text{QED})$$