ROMANIAN MATHEMATICAL MAGAZINE

If $x\sqrt{y^2+2}+y\sqrt{x^2+2}=4\sqrt{6}$ **then prove that :**

$$\sqrt{x^2+2}+\sqrt{y^2+2}+x+y\ge 2\sqrt{6}+4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(x.\sqrt{y^2+2}\right)^2=\left(4\sqrt{6}-y.\sqrt{x^2+2}\right)^2$$

$$\Rightarrow x^2(y^2+2)=96+x^2y^2+2y^2-8y.\sqrt{6x^2+12}$$

$$\Rightarrow y^2-4y.\sqrt{6x^2+12}+48-x^2=0$$

$$\Rightarrow y=\frac{4.\sqrt{6x^2+12}\pm\sqrt{96x^2+192-192+4x^2}}{2}$$

$$\Rightarrow y=2.\sqrt{6x^2+12}+5x \text{ or } y=2.\sqrt{6x^2+12}-5x$$

$\boxed{\text{Case 1}}$ If $y=2.\sqrt{6x^2+12}+5x$, then :

$$x.\sqrt{y^2+2}+\left(2.\sqrt{6x^2+12}+5x\right).\sqrt{x^2+2}=4\sqrt{6}$$

$$\Rightarrow x.\sqrt{y^2+2}+2\sqrt{6}.(x^2+2)+5x.\sqrt{x^2+2}=4\sqrt{6}$$

$$\Rightarrow x\left(\sqrt{y^2+2}+2\sqrt{6}.x+5.\sqrt{x^2+2}\right)\overset{(*)}{=}0 \text{ and when } x\ge 0,$$

$$\sqrt{y^2+2}+2\sqrt{6}.x+5.\sqrt{x^2+2} \text{ is trivially} >0 \text{ and when } x<0,$$

$$\sqrt{y^2+2}+2\sqrt{6}.x+5.\sqrt{x^2+2}=\sqrt{y^2+2}+\frac{25x^2+50-24x^2}{5.\sqrt{x^2+2}-2\sqrt{6}.x}$$

$$=\sqrt{y^2+2}+\frac{x^2+50}{5.\sqrt{x^2+2}-2\sqrt{6}.x}>0 \text{ and so}, \sqrt{y^2+2}+2\sqrt{6}.x+5.\sqrt{x^2+2}$$

$$>0\ \forall\ x,y\in\mathbb{R} \text{ and so}, (*)\Rightarrow x=0 \text{ and then : } y=4\sqrt{3}\Rightarrow x+y=4\sqrt{3}>4$$

$\boxed{\text{Case 2}}$ If $y=2.\sqrt{6x^2+12}-5x$, then : $x+y=2.\sqrt{6x^2+12}-4x=t\,(say)$

$$\Rightarrow 24x^2+48=16x^2+t^2+8xt\Rightarrow 8x^2-8t.x+48-t^2=0 \text{ and}$$

$$\because x\in\mathbb{R}, \text{discriminant}\ge 0\Rightarrow 64t^2-32(48-t^2)\ge 0\Rightarrow t^2\ge 16$$

$$\Rightarrow \boxed{x+y\ge 4 \text{ or } x+y\le -4} \text{ and if } t=x+y\le -4, \text{ then : } 2.\sqrt{6x^2+12}-4x=t$$

$$\le -4<0\Rightarrow\sqrt{6x^2+12}<2x\Rightarrow 6x^2+12<4x^2\Rightarrow 2x^2+12<0\to\text{impossible}$$

$$\therefore x+y\ge 4 \text{ and so, combining both cases}, x.\sqrt{y^2+2}+y.\sqrt{x^2+2}=4\sqrt{6}$$

$$\Rightarrow x+y\geq 4 \to (1) \text{ and now}, \sqrt{x^2+2}+x \overset{?}{\geq} 2+\sqrt{6}+\frac{(x-2)(2+\sqrt{6})}{\sqrt{6}}$$

$$\Leftrightarrow \sqrt{x^2+2}-\sqrt{6} \overset{?}{\geq} (x-2)\left(\frac{2+\sqrt{6}}{\sqrt{6}}-1\right) \Leftrightarrow \sqrt{6x^2+12}-6 \overset{?}{\geq} 2x-4$$

$$\Leftrightarrow \sqrt{6x^2+12} \overset{?}{\geq} 2x+2 \text{ and it's trivially true when}: x\leq -1 \text{ and when}:$$

$$x>-1, \sqrt{6x^2+12} \overset{?}{\geq} 2x+2 \Leftrightarrow 6x^2+12 \overset{?}{\geq} 4x^2+4+8x \Leftrightarrow x^2-4x+4 \overset{?}{\geq} 0$$

$$\Leftrightarrow (x-2)^2 \overset{?}{\geq} 0 \to \text{true} \therefore \sqrt{x^2+2}+x \geq 2+\sqrt{6}+\frac{(x-2)(2+\sqrt{6})}{\sqrt{6}}$$

$$\forall x\in\mathbb{R} \text{ and analogously}, \sqrt{y^2+2}+y \geq 2+\sqrt{6}+\frac{(y-2)(2+\sqrt{6})}{\sqrt{6}} \text{ and so},$$

$$\sqrt{x^2+2}+x+\sqrt{y^2+2}+y \geq$$

$$2+\sqrt{6}+\frac{(x-2)(2+\sqrt{6})}{\sqrt{6}}+2+\sqrt{6}+\frac{(y-2)(2+\sqrt{6})}{\sqrt{6}}$$

$$= 2\sqrt{6}+4+\frac{(x+y-4)(2+\sqrt{6})}{\sqrt{6}} \overset{\text{via } (1)}{\geq} 2\sqrt{6}+4 \therefore x.\sqrt{y^2+2}+y.\sqrt{x^2+2}=4\sqrt{6}$$

$$\Rightarrow \sqrt{x^2+2}+\sqrt{y^2+2}+x+y \geq 2\sqrt{6}+4, \text{"="} \text{ iff } x=y=2 \text{ (QED)}$$