ROMANIAN MATHEMATICAL MAGAZINE

If $x\sqrt{y^2+2}+y\sqrt{x^2+2}=4\sqrt{6}$ then prove that :

$$x+y\geq 4$$

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$$\left(x.\sqrt{y^2+2}\right)^2=\left(4\sqrt{6}-y.\sqrt{x^2+2}\right)^2$$

$$\Rightarrow x^2(y^2+2)=96+x^2y^2+2y^2-8y.\sqrt{6x^2+12}$$

$$\Rightarrow y^2-4y.\sqrt{6x^2+12}+48-x^2=0$$

$$\Rightarrow y=\frac{4.\sqrt{6x^2+12}\pm\sqrt{96x^2+192-192+4x^2}}{2}$$

$$\Rightarrow y=2.\sqrt{6x^2+12}+5x \text{ or } y=2.\sqrt{6x^2+12}-5x$$

$$\boxed{\text{Case 1}}\ \text{If } y=2.\sqrt{6x^2+12}+5x, \text{then :}$$

$$x.\sqrt{y^2+2}+\left(2.\sqrt{6x^2+12}+5x\right).\sqrt{x^2+2}=4\sqrt{6}$$

$$\Rightarrow x.\sqrt{y^2+2}+2\sqrt{6}.(x^2+2)+5x.\sqrt{x^2+2}=4\sqrt{6}$$

$$\Rightarrow x\left(\sqrt{y^2+2}+2\sqrt{6}.x+5.\sqrt{x^2+2}\right)\overset{(*)}{=}0 \text{ and when } x\geq 0,$$

$$\sqrt{y^2+2}+2\sqrt{6}.x+5.\sqrt{x^2+2} \text{ is trivially} > 0 \text{ and when } x<0,$$

$$\sqrt{y^2+2}+2\sqrt{6}.x+5.\sqrt{x^2+2}=\sqrt{y^2+2}+\frac{25x^2+50-24x^2}{5.\sqrt{x^2+2}-2\sqrt{6}.x}$$

$$=\sqrt{y^2+2}+\frac{x^2+50}{5.\sqrt{x^2+2}-2\sqrt{6}.x}>0 \text{ and so, } \sqrt{y^2+2}+2\sqrt{6}.x+5.\sqrt{x^2+2}$$

$$>0\ \forall\, x,y\in\mathbb{R} \text{ and so, } (*)\Rightarrow x=0 \text{ and then : } y=4\sqrt{3}\Rightarrow x+y=4\sqrt{3}>4$$

$$\boxed{\text{Case 2}}\ \text{If } y=2.\sqrt{6x^2+12}-5x, \text{then : } x+y=2.\sqrt{6x^2+12}-4x=t\ (say)$$

$$\Rightarrow 24x^2+48=16x^2+t^2+8xt\Rightarrow 8x^2-8t.x+48-t^2=0 \text{ and}$$

$$\because x\in\mathbb{R}, \text{discriminant}\geq 0\Rightarrow 64t^2-32(48-t^2)\geq 0\Rightarrow t^2\geq 16$$

$$\Rightarrow \boxed{x+y\geq 4 \text{ or } x+y\leq -4} \text{ and if } t=x+y\leq -4, \text{then : } 2.\sqrt{6x^2+12}-4x=t$$

$$\leq -4<0\Rightarrow\sqrt{6x^2+12}<2x\Rightarrow 6x^2+12<4x^2\Rightarrow 2x^2+12<0\rightarrow \text{impossible}$$

$\therefore\ x+\mathrm{y}\geq 4$ and so, combining both $cases$, $x.\sqrt{\mathrm{y}^2+2}+\mathrm{y}.\sqrt{x^2+2}=4\sqrt{6}\Rightarrow$

$x+\mathrm{y}\geq 4$ and $x+\mathrm{y}=4\Rightarrow 2.\sqrt{6x^2+12}-4x=4\Rightarrow x=2\Rightarrow \mathrm{y}=2$ and so,

$''=''$ iff $x=\mathrm{y}=2$ (QED)