ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, $x^2 + y^2 + z^2 = 3$ then:

$$\sqrt{8x^2 + 14xy + 3y^2} + \sqrt{8y^2 + 14zy + 3z^2} + \sqrt{8z^2 + 14xz + 3x^2} \le 15$$

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Lemma:

$$\forall a, b > 0, \sqrt{8a^2 + 14ab + 3b^2} \le 3a + 2b$$

Proof:

$$\textit{We need to show}:$$

$$\sqrt{8a^2 + 14ab + 3b^2} \le 3a + 2b$$

$$8a^2 + 14ab + 3b^2 \overset{squaring}{\le} (3a + 2b)^2$$

$$8a^2 + 14ab + 3b^2 \le 9a^2 + 12ab + 4b^2$$

$$a^2 - 2ab + b^2 \ge 0 \text{ or}, (a - b)^2 \ge 0 \text{ true}$$

$$\sqrt{8x^2 + 14xy + 3y^2} + \sqrt{8y^2 + 14zy + 3z^2} + \sqrt{8z^2 + 14xz + 3x^2} =$$

$$= \sum \sqrt{8x^2 + 14xy + 3y^2} \le$$

$$\overset{Lemma}{\le} \sum (3x + 2y) = 5(x + y + z) = 5\sqrt{(x + y + z)^2} \le$$

$$\le 5\sqrt{3(x^2 + y^2 + z^2)} \overset{x^2+y^2+z^2=3}{\le} 5 \times 3 = 15$$

Equality holds for x=y=z=1.