ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, $xy + yz + zx \geq 3$ then:

$$\frac{x^2}{\sqrt{8x^2 + 14xy + 3y^2}} + \frac{y^2}{\sqrt{8y^2 + 14zy + 3z^2}} + \frac{z^2}{\sqrt{8z^2 + 14xz + 3x^2}} \geq \frac{3}{5}$$

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Lemma:

$$\forall a, b > 0 \ \sqrt{8a^2 + 14ab + 3b^2} \leq 3a + 2b$$

Proof:

$$\textit{We need to show}:$$

$$\sqrt{8a^2 + 14ab + 3b^2} \leq 3a + 2b$$

$$8a^2 + 14ab + 3b^2 \overset{squaring}{\leq} (3a + 2b)^2$$

$$8a^2 + 14ab + 3b^2 \leq 9a^2 + 12ab + 4b^2$$

$$a^2 - 2ab + b^2 \geq 0 \text{ or } (a - b)^2 \geq 0 \text{ true}$$

$$\frac{x^2}{\sqrt{8x^2 + 14xy + 3y^2}} + \frac{y^2}{\sqrt{8y^2 + 14zy + 3z^2}} + \frac{z^2}{\sqrt{8z^2 + 14xz + 3x^2}} =$$

$$= \sum \frac{x^2}{\sqrt{8x^2 + 14xy + 3y^2}} \overset{Lemma}{\geq} \sum \frac{x^2}{3x + 2y} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(x + y + z)^2}{3(x + y + z) + 2(x + y + z)} = \frac{x + y + z}{5} =$$

$$= \frac{\sqrt{(x + y + z)^2}}{5} \geq \frac{\sqrt{3(xy + yz + zx)}}{5} \overset{xy+yz+zx\geq 3}{\geq} \frac{3}{5}$$

$$\textit{Equality holds for } x = y = z = 1$$