ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \le 2025$ then:

$$\frac{1}{\sqrt{7x^2-2xy+4y^2}}+\frac{1}{\sqrt{7y^2-2yz+4z^2}}+\frac{1}{\sqrt{7z^2-2zx+4x^2}} \le 675$$

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Lemma:

$$\forall\, a, b > 0, \sqrt{7a^2-2ab+4b^2} \ge 2a+b$$

Proof:

$$\textit{We need to show}:$$

$$\sqrt{7a^2-2ab+4b^2} \ge 2a+b$$

$$7a^2-2ab+4b^2 \overset{squaring}{\ge} (2a+b)^2$$

$$7a^2-2ab+4b^2 \overset{squaring}{\ge} 4a^2+4ab+b^2$$

$$3a^2-6ab+3b^2 \ge 0 \text{ or } 3(a-b)^2 \ge 0 \text{ true}$$

$$\frac{1}{\sqrt{7x^2-2xy+4y^2}}+\frac{1}{\sqrt{7y^2-2yz+4z^2}}+\frac{1}{\sqrt{7z^2-2zx+4x^2}} =$$

$$= \sum \frac{1}{\sqrt{7x^2-2xy+4y^2}} \overset{lemma}{\le} \sum \frac{1}{2x+y} = \sum \frac{1}{x+x+y} \overset{AM-HM}{\le}$$

$$\le \frac{1}{9}\sum\left(\frac{1}{x}+\frac{1}{x}+\frac{1}{y}\right) = \frac{3}{9}\sum\frac{1}{x} \overset{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\le 2025}{\le} \frac{1}{3}.2025 = 675$$

$$\textit{Equality holds for } x=y=z=\frac{1}{675}$$