ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0,\ a + b = 2$ then:

$$ab + \frac{1}{a^2 + b^2 + 1} - \frac{1}{a^3 + b^3 + 1} + \frac{1}{a^4 + b^4 + 1} \le \frac{4}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$ab \overset{AM-GM}{\le} \left(\frac{a+b}{2}\right)^2 \overset{a+b=2}{=} 1\ (1)$$

$$a^2 + b^2 = \frac{a^2}{1} + \frac{b^2}{1} \overset{Radon}{\ge} \frac{(a+b)^2}{2} \overset{a+b=2}{=} 2\ \ (2)$$

$$\frac{a^4}{1^3} + \frac{b^4}{1^3} \overset{Radon}{\ge} \frac{(a+b)^4}{8} \overset{a+b=2}{=} 2\ \ (3)$$

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b) \overset{a+b=2}{=} 8 - 6ab\ (4)$$

$$We\ need\ to\ show:$$

$$ab + \frac{1}{a^2 + b^2 + 1} - \frac{1}{a^3 + b^3 + 1} + \frac{1}{a^4 + b^4 + 1} \le \frac{4}{3}$$

$$ab + \frac{1}{2+1} - \frac{1}{9 - 6ab} + \frac{1}{2+1} \overset{(2),(3)\&(4)}{\le} \frac{4}{3}$$

$$ab - \frac{1}{9 - 6ab} \le \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

$$x - \frac{1}{9 - 6x} \overset{ab=x}{\le} \frac{2}{3}\ or\ \frac{9x - 6x^2 - 1}{9 - 6x} \le \frac{2}{3}$$

$$27x - 18x^2 - 3 \le 18 - 12x\ or\ 18x^2 - 39x + 21 \ge 0$$

$$(x-1)(6x-7) \ge 0\ \ true\ as\ x = ab \le 1\ \big(by\ (1)\big)\ then$$

$$(x-1) \le 0\ and\ (6x-7) < 0\ so\ (x-1)(6x-7) \ge 0$$

Equality holds for a=b=1.