ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, then prove that :

$$\frac{4a^2+(b-c)^2}{2a^2+b^2+c^2}+\frac{4b^2+(c-a)^2}{2b^2+c^2+a^2}+\frac{4c^2+(a-b)^2}{2c^2+a^2+b^2}\geq 3$$

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$$\sum_{cyc}\frac{4a^2+(b-c)^2}{2a^2+b^2+c^2}=\sum_{cyc}\frac{2a^2+b^2+c^2}{2a^2+b^2+c^2}+\sum_{cyc}\frac{2(a^2-bc)}{2a^2+b^2+c^2}\geq 3$$

$$\Leftrightarrow\sum_{cyc}\left((a^2-bc)\left(b^2+\sum_{cyc}a^2\right)\left(c^2+\sum_{cyc}a^2\right)\right)\geq 0$$

$$\Leftrightarrow\sum_{cyc}\left((a^2-bc)\left(b^2c^2+\left(\sum_{cyc}a^2\right)^2+\left(\sum_{cyc}a^2\right)\left(-a^2+\sum_{cyc}a^2\right)\right)\right)\geq 0$$

$$\Leftrightarrow\sum_{cyc}\left((a^2-bc)\left(b^2c^2+2\left(\sum_{cyc}a^2\right)^2-a^2\left(\sum_{cyc}a^2\right)\right)\right)\geq 0$$

$$\Leftrightarrow 2\left(\sum_{cyc}a^2\right)^2\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)+3a^2b^2c^2-\sum_{cyc}a^3b^3-$$

$$\left(\sum_{cyc}a^2\right)\left(\left(\sum_{cyc}a^2\right)^2-2\sum_{cyc}a^2b^2-abc\left(\sum_{cyc}a\right)\right)\geq 0$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^3-2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)^2+2\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)+$$

$$abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)+3a^2b^2c^2-\sum_{cyc}a^3b^3\ \boxed{\overset{(*)}{\geq}}\ 0$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius

$= s, R, r\ (say)$; then : $\sum_{cyc} a = s \to (1), abc = r^2 s \to (2), \sum_{cyc} ab = 4Rr + r^2 \to (3),$

$$\sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4), \sum_{cyc} a^2b^2 = r^2((4R+r)^2 - 2s^2) \to (5),$$

$$\sum_{cyc} a^3b^3 = (4Rr + r^2)^3 - 12Rr^3s^2 \to (6) \therefore (*) \overset{via\ (1),(2),(3),(4),(5)\ and\ (6)}{\Leftrightarrow}$$

$$(s^2 - 8Rr - 2r^2)^3 - 2(4Rr + r^2)(s^2 - 8Rr - 2r^2)^2 +$$
$$2r^2((4R+r)^2 - 2s^2)(s^2 - 8Rr - 2r^2) + r^2s^2(s^2 - 8Rr - 2r^2) +$$
$$3r^4s^2 - (4Rr + r^2)^3 + 12Rr^3s^2 \geq 0 \Leftrightarrow$$

$$s^6 - (32Rr + 11r^2)s^4 + r^2(352R^2 + 212Rr + 31r^2)s^2 - 21r^3(4R+r)^3 \boxed{\overset{(**)}{\geq}} 0\ and$$

$$\because P = (s^2 - 16Rr + 5r^2)^3 + 2r(8R - 13r)(s^2 - 16Rr + 5r^2)^2 +$$

$$12r^2(8R^2 - 25Rr + 18r^2)(s^2 - 16Rr + 5r^2) \overset{Gerretsen}{\geq} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**) \overset{?}{\geq} P \Leftrightarrow 4t^3 - 19t^2 + 28t - 12 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (4t-3)(t-2)^2 \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{4a^2 + (b-c)^2}{2a^2 + b^2 + c^2} + \frac{4b^2 + (c-a)^2}{2b^2 + c^2 + a^2} + \frac{4c^2 + (a-b)^2}{2c^2 + a^2 + b^2} \geq 3\ \forall\ a, b, c > 0,$$
$$'' ='' \text{ iff } a = b = c\ (QED)$$