ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a + b + c + 2 = abc$, then prove that :

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \le \frac{3}{2}\sqrt{abc}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } a = x + 1, b = y + 1, c = z + 1 \text{ and then :}$$

$$x + y + z + 5 = (x + 1)(y + 1)(z + 1) \Rightarrow xyz + xy + yz + zx = 4$$

$$\Rightarrow \sum_{\text{cyc}} \big((2 + y)(2 + z)\big) - (2 + x)(2 + y)(2 + z) = 4 - (xy + yz + zx + xyz) = 0$$

$$\therefore \sum_{\text{cyc}} \frac{1}{2 + x} = 1 \to (\text{m}) \ (\because 2 + x = a + 1 > 1 > 0 \text{ and analogs}) \text{ and setting :}$$

$$\frac{1}{2 + x} = \frac{1}{2} - \alpha \Rightarrow x = \frac{2\alpha}{\frac{1}{2} - \alpha} \to (1); \text{ similarly, setting : } \frac{1}{2 + y} = \frac{1}{2} - \beta \text{ and}$$

$$\frac{1}{2 + c} = \frac{1}{2} - \gamma \Rightarrow 1 \overset{\text{via (m)}}{=} \frac{1}{2} - \alpha + \frac{1}{2} - \beta + \frac{1}{2} - \gamma \Rightarrow \alpha + \beta + \gamma = \frac{1}{2} \to (\text{i})$$

$$\text{We note that : } \beta + \gamma = \frac{1}{2} - \alpha = \frac{1}{2 + x} > 0 \text{ and similarly, } \gamma + \alpha > 0 \text{ and } \alpha + \beta > 0$$

$$\therefore (1) \text{ and (i)} \Rightarrow x = \frac{2\alpha}{\beta + \gamma} \text{ and analogously, } y = \frac{2\beta}{\gamma + \alpha} \text{ and } z = \frac{2\gamma}{\alpha + \beta}$$

$$\text{and via such substitutions, } \sqrt{a} + \sqrt{b} + \sqrt{c} \le \frac{3}{2}.\sqrt{abc}$$

$$\Leftrightarrow \sqrt{\frac{2\alpha}{\beta + \gamma} + 1} + \sqrt{\frac{2\beta}{\gamma + \alpha} + 1} + \sqrt{\frac{2\gamma}{\alpha + \beta} + 1} \le$$

$$\frac{3}{2}.\sqrt{\left(\frac{2\alpha}{\beta + \gamma} + 1\right)\left(\frac{2\beta}{\gamma + \alpha} + 1\right)\left(\frac{2\gamma}{\alpha + \beta} + 1\right)}$$

$$\Leftrightarrow \sqrt{\frac{Y + Z}{X}} + \sqrt{\frac{Z + X}{Y}} + \sqrt{\frac{X + Y}{Z}} \boxed{\overset{(*)}{\le}} \frac{3}{2}.\sqrt{\frac{(Y + Z)(Z + X)(X + Y)}{XYZ}}$$

$$(X = \beta + \gamma > 0; Y = \gamma + \alpha > 0; Z = \alpha + \beta > 0)$$

$$\text{Now, } \sum_{\text{cyc}} \sqrt{\frac{Y + Z}{X}} \overset{\text{CBS}}{\le} \sqrt{\sum_{\text{cyc}} (Y + Z)}.\sqrt{\sum_{\text{cyc}} \frac{1}{X}} = \frac{3}{2}.\sqrt{\frac{8(\sum_{\text{cyc}} X)(\sum_{\text{cyc}} XY)}{9XYZ}}$$

$$= \frac{3}{2}.\sqrt{\frac{9(\sum_{\text{cyc}} X)(\sum_{\text{cyc}} XY) - (\sum_{\text{cyc}} X)(\sum_{\text{cyc}} XY)}{9XYZ}} \overset{\text{AM-GM}}{\le} \frac{3}{2}.\sqrt{\frac{9(\sum_{\text{cyc}} X)(\sum_{\text{cyc}} XY) - 9XYZ}{9XYZ}}$$

$$=\frac{3}{2}\cdot\sqrt{\frac{(Y+Z)(Z+X)(X+Y)}{XYZ}}\Rightarrow(*)\text{ is true}\Rightarrow\sqrt{a}+\sqrt{b}+\sqrt{c}\le\frac{3}{2}\cdot\sqrt{abc}$$

$$\forall\, a,b,c>0\,\middle|\,a+b+c+2=abc,''=''\text{ iff } x=y=z=1$$

$$\Rightarrow ''=''\text{ iff } a=b=c=2\ (\text{QED})$$