ROMANIAN MATHEMATICAL MAGAZINE

If $a+b+c+2=abc$ then:

$$\frac{1}{\sqrt{7+a}}+\frac{1}{\sqrt{7+b}}+\frac{1}{\sqrt{7+c}}\le 1$$

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$$If\ \ a+b+c+2=abc\ then\ \exists\ x,y,z>0\ such\ that$$
$$a=\frac{x+y}{z},b=\frac{y+z}{x},c=\frac{x+z}{y},$$
$$We\ take\ f(a)=\frac{6a}{6a+s}\ where\ a>0\ ,f''(a)=-\frac{72}{(6a+s)^3}<0,$$
$$so\ f\ is\ concave\ for\ a>0.Using\ Jensen\ inequality\ we\ get:$$

$$\sum\frac{6z}{6z+s}s=f(z)+f(y)+f(x)\ge 3f\left(\frac{x+y+z}{3}\right)\overset{s=x+y+z}{=}3f\left(\frac{s}{3}\right)=$$
$$=3\frac{6\times\frac{s}{3}}{6\times\frac{s}{3}+s}s=2\ (1)$$

$$\frac{1}{\sqrt{7+a}}+\frac{1}{\sqrt{7+b}}+\frac{1}{\sqrt{7+c}}=$$
$$=\sum\frac{1}{\sqrt{7+a}}=\sum\frac{1}{\sqrt{7+\frac{x+y}{z}}}=\sum\sqrt{\frac{z}{6z+(x+y+z)}}\overset{s=x+y+z}{=}$$
$$=\sum\sqrt{\frac{z}{6z+s}}\overset{CBS}{\le}\sqrt{3\left(\sum\frac{z}{6z+s}\right)}=\sqrt{\frac{3}{6}\left(\sum\frac{6z}{6z+s}\right)}=$$
$$=\sqrt{\frac{1}{2}\left(\sum\frac{z}{6z+s}\right)}\overset{(1)}{\le}\sqrt{\frac{1}{2}}\times 2=1$$

Equality holds for a=b=c=2.