ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ and $a^4 + b^4 = ab(a+b)$ then:

$$a^2 + b^2 \le 2$$

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$$a^4 + b^4 = \frac{a^4}{1^3} + \frac{b^4}{1^3} \overset{Radon}{\ge} \frac{(a+b)^4}{8} \ (1)$$

$$ab(a+b) \overset{AM-GM}{\le} \frac{(a+b)^2}{4}(a+b) = \frac{(a+b)^3}{4} \ (2)$$

$$a^4 + b^4 = ab(a+b)\,.Using\ (1)\&\ (2) we\ get:$$
$$\frac{(a+b)^4}{8} \le \frac{(a+b)^3}{4}\ or, (a+b) \le 2\ \ (3)$$
$$We\ need\ to\ show:$$
$$a^2 + b^2 \le 2\ \ or\ (a+b)^2 - 2ab \le 2$$

$$(a+b)^2 - \frac{2(a^4+b^4)}{a+b} \overset{a^4+b^4=ab(a+b)}{\le} 2$$
$$(a+b)^2 - \frac{2(a+b)^3}{8} \overset{(1)}{\le} 2\ or\ \ 4(a+b)^2 - (a+b)^3 \le 8$$
$$x^3 - 4x^2 + 8 \overset{x=a+b}{\ge} 0\ or\ (x-2)(x^2 - 2x - 4) \ge 0$$

$$(x-2)(x(x-2)-4) \ge 0\ true\ since\ by\ (3)\ x = a+b \le 2$$

$$so\ (x-2) < 0\ and\ (x(x-2)-4) < 0\ and (x-2)(x(x-2)-4) \ge 0$$

Equality holds for a=b=1.