ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $xy + yz + zx + 2xyz = 1$ then prove that:

$$\sqrt{1-x^2} + \sqrt{1-y^2} + \sqrt{1-z^2} \le \frac{3\sqrt{3}}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$1 = 2xyz + \sum_{cyc} xy \overset{AM-GM}{\ge} 2t^3 + 3t^2 \left(t = \sqrt[3]{xyz}\right) \Rightarrow (2t-1)(t+1)^2 \le 0$$

$$\Rightarrow t \le \frac{1}{2} \to (1) \text{ and } \sum_{cyc} \sqrt{1-x^2} \overset{CBS}{\le} \sqrt{3\left(3 - \sum_{cyc} x^2\right)} \le \sqrt{3\left(3 - \sum_{cyc} xy\right)}$$

$$\overset{xy+yz+zx+2xyz=1}{=} \sqrt{3(3-1+2xyz)} \overset{t \le \frac{1}{2}}{\le} \sqrt{3\left(2+\frac{2}{8}\right)} = \frac{3\sqrt{3}}{2}$$

$$\therefore \sqrt{1-x^2} + \sqrt{1-y^2} + \sqrt{1-z^2} \le \frac{3\sqrt{3}}{2} \ \forall\, x, y, z > 0 \,\middle|\, xy + yz + zx + 2xyz = 1,$$

$$'' ='' \text{ iff } x = y = z = \frac{1}{2} \text{ (QED)}$$