ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0,\ a^5 + b^5 \le 2a^3b^3$ then:

$$\frac{1}{a^2} + \frac{1}{b^2} \le 2$$

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$$a^5 + b^5 \le 2a^3b^3 \ \ or\ \ 2(ab)^{\frac{5}{2}} \overset{AM-GM}{\le} 2a^3b^3\ or$$

$$1 \le \frac{(ab)^3}{(ab)^{\frac{5}{2}}}\ or, \sqrt{ab} \ge 1\ or, ab \ge 1\ \ (1)$$

$$a^5 + b^5 \overset{Chebyshev}{\ge} \frac{1}{2}(a^3 + b^3)(a^2 + b^2) \overset{AM-GM}{\ge}$$

$$\ge \frac{1}{2}2\sqrt{a^3b^3}(a^2 + b^2) = \sqrt{a^3b^3}(a^2 + b^2)\ (2)$$

$$According\ to\ the\ given\ condition:$$

$$a^5 + b^5 \le 2a^3b^3\ or \sqrt{a^3b^3}(a^2 + b^2) \overset{(2)}{\le} 2a^3b^3\ or$$

$$\frac{a^2 + b^2}{a^2b^2} \le \frac{2ab}{\sqrt{a^3b^3}} = \frac{2}{\sqrt{ab}} \overset{(1)}{\le} \frac{2}{1} = 2$$

Equality holds for a=b=1.