ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$,
$a^2(1+b)(1+c)+b^2(1+c)(1+a)+c^2(1+a)(1+b)\ge 12$ then:

$$a^2+b^2+c^2\ge 3$$

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$$WLOG\ a\ge b\ge c\ then\ (1+b)(1+c)\le(1+c)(1+a)\le(1+a)(1+b)\ then$$

$$a^2(1+b)(1+c)+b^2(1+c)(1+a)+c^2(1+a)(1+b)\le$$
$$\overset{Chebyshev}{\le}\frac{1}{3}(a^2+b^2+c^2)\big((1+b)(1+c)+(1+c)(1+a)+(1+a)(1+b)\big)=$$
$$=\frac{1}{3}(a^2+b^2+c^2)\big(3+2(a+b+c)+(ab+bc+ca)\big)=$$
$$=\frac{1}{3}(a^2+b^2+c^2)\left(3+2\sqrt{(a+b+c)^2}+(ab+bc+ca)\right)=$$
$$\le\frac{1}{3}(a^2+b^2+c^2)\left(3+2\sqrt{3(a^2+b^2+c^2)}+(a^2+b^2+c^2)\right)=$$
$$\overset{a^2+b^2+c^2=x^2}{=}\frac{1}{3}\left(3x^2+2\sqrt{3}x^3+x^4\right)\ (1)$$

$$According\ to\ the\ given\ condition:$$
$$a^2(1+b)(1+c)+b^2(1+c)(1+a)+c^2(1+a)(1+b)\ge 12$$
$$\frac{1}{3}\left(3x^2+2\sqrt{3}x^3+x^4\right)\overset{(1)}{\ge}12\ or\ \left(3x^2+2\sqrt{3}x^3+x^4\right)-36\ge 0$$
$$\left(x-\sqrt{3}\right)\left(x^3+3\sqrt{3}x^2+12x+12\sqrt{3}\right)\ge 0\ now\ clearly\ \left(x-\sqrt{3}\right)\ge 0$$
$$or\ x\ge\sqrt{3}\ or\ x^2=a^2+b^2+c^2\ge 3$$

Equality holds for a=b=c=1.