ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, $xy + yz + zx + xyz = 4$ then:

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{x+y+z}{2}$$

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$$xy + yz + zx + xyz = 4 \text{ can be written as}$$

$$\frac{x}{x+2} + \frac{y}{y+2} + \frac{z}{z+2} = 1$$

$$1 = \sum \frac{x}{x+2} = \sum \frac{x^2}{x^2+2x} \overset{CBS}{\geq} \frac{(x+y+z)^2}{x^2+y^2+z^2+2x+2y+2z}$$

$$x^2 + y^2 + z^2 + 2(x+y+z) \geq (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy+yz+zx)$$

$$x + y + z \geq xy + yz + zx \quad (1)$$

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = \sum \frac{x}{y+z} = \sum \frac{x^2}{yx+xz} \overset{CBS}{\geq} \frac{(x+y+z)^2}{2(xy+yz+zx)} \overset{(1)}{\geq} \frac{x+y+z}{2}$$

Equality holds for x=y=z=1.