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If $a, b, c > 0$ then prove that :

$$\sum_{cyc} \frac{(b+c)(b^2+ca)}{b^2+bc+c^2} \geq \frac{4(ab+bc+ca)}{a+b+c}$$

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We have $\forall\, a, b, c, u, v, w > 0, (b+c)u + (c+a)v + (a+b)w \geq 2\sqrt{(ab+bc+ca)(uv+vw+wu)} \to (1)$ and choosing :

$u = \frac{b^2+ca}{b^2+bc+c^2}, v = \frac{c^2+ab}{c^2+ca+a^2}, w = \frac{a^2+bc}{a^2+ab+b^2}$ in (1), we get :

$$\frac{(b+c)(b^2+ca)}{b^2+bc+c^2} + \frac{(c+a)(c^2+ab)}{c^2+ca+a^2} + \frac{(a+b)(a^2+bc)}{a^2+ab+b^2} \geq$$

$$2\sqrt{\left(\sum_{cyc} ab\right) . \sum_{cyc} \frac{(b^2+ca)(c^2+ab)}{(b^2+bc+c^2)(c^2+ca+a^2)}}$$

$$= 2\sqrt{\left(\sum_{cyc} ab\right) . \frac{\sum_{cyc} ab^5 + abc\sum_{cyc} a^2b + \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) + abc\sum_{cyc} a^3 + 2\sum_{cyc} a^3b^3 + 2abc\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc\right)}{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) + abc\sum_{cyc} a^3 + \sum_{cyc} a^3b^3 + 2abc\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc\right)}} \overset{AM-GM}{\geq}$$

$$2\sqrt{\left(\sum_{cyc} ab\right) . \frac{3a^2b^2c^2 + 3a^2b^2c^2 + \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) + abc\sum_{cyc} a^3 + 2\sum_{cyc} a^3b^3 + 2abc\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc\right)}{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) + abc\sum_{cyc} a^3 + \sum_{cyc} a^3b^3 + 2abc\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc\right)}} \overset{?}{\geq} \frac{4(ab+bc+ca)}{a+b+c}$$

$$\Leftrightarrow \left(\sum_{cyc} a\right)^2 . \left(\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) + abc\sum_{cyc} a^3 + 2\sum_{cyc} a^3b^3 + 2abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)\right) \overset{?}{\geq} (*)$$

$$4\left(\sum_{cyc} ab\right)\left(\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) + abc\sum_{cyc} a^3 + \sum_{cyc} a^3b^3\right) +$$

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$$4\left(\sum_{cyc} ab\right)\left(2abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)-6a^2b^2c^2\right) \text{ \& assigning } b+c=x, c+a=y,$$
$$a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0 \text{ and } z+x-y=2b>0$$
$$\Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z \text{ form sides of a triangle XYZ with}$$
$$\text{semiperimeter, circumradius \& inradius} = s, R, r \text{ (say); then}: \sum_{cyc} a = s, abc = r^2 s$$
$$\sum_{cyc} ab = 4Rr + r^2, \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2, \sum_{cyc} a^3 = s^3 - 12Rrs,$$
$$\sum_{cyc} a^2b^2 = r^2((4R+r)^2 - 2s^2) \text{ \& } \sum_{cyc} a^3b^3 = (4Rr+r^2)^3 - 12Rr^3s^2$$
$$\text{\& with these, } (*) \text{ post simplification gets transformed into}:$$
$$-s^6 + (16R^2+12Rr+11r^2)s^4 - 4r(4R+r)^3.s^2 + 4r^2(4R+r)^4 \boxed{\overset{?}{\underset{(**)}{\geq}}} 0 \text{ and } \because P =$$
$$-s^2(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)+$$
$$(12R^2-8Rr+13r^2)(s^2-16Rr+5r^2)^2$$
$$+r(192R^3-520R^2r+460Rr^2-133r^3)(s^2-16Rr+5r^2) \overset{\substack{\text{Double-Rouche}\\ \text{and}\\ \text{Gerretsen}}}{\geq} 0$$
$$\left(\because 192R^3-520R^2r+460Rr^2-133r^3 = (R-2r)(192R^2-136Rr+188r^2)+243r^3 \overset{\text{Euler}}{\geq} 243r^3 > 0\right)$$
$$\therefore \text{ in order to prove } (**), \text{ it suffices to prove}: \text{LHS of } (**) \overset{?}{\geq} P$$
$$\Leftrightarrow 256t^4 - 1072t^3 + 1359t^2 - 521t + 86 \overset{?}{\geq} 0 \quad \left(t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\Big((t-2)(256t^2-48t+143)+243\Big) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow$$
$$(*) \text{ is true} \therefore \frac{(b+c)(b^2+ca)}{b^2+bc+c^2} + \frac{(c+a)(c^2+ab)}{c^2+ca+a^2} + \frac{(a+b)(a^2+bc)}{a^2+ab+b^2} \geq$$
$$\frac{4(ab+bc+ca)}{a+b+c} \ \forall a,b,c>0, \text{"="} \text{ iff } a=b=c \text{ (QED)}$$