ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $ab + bc + ca = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ then prove that :

$$\frac{1}{5a + 4b + 3c + 12} + \frac{1}{5b + 4c + 3a + 12} + \frac{1}{5c + 4a + 3b + 12} \le \frac{1}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$ab + bc + ca = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \Rightarrow abc = 1 \Rightarrow \sum_{cyc} a \overset{AM-GM}{\ge} 3.\sqrt[3]{abc} = 3 \to (1)$$

$$\Rightarrow 5a + 4b + 3c + 12 = 3\sum_{cyc} a + 2a + b + 12 \ge 9 + 2a + b + 12$$

$$\Rightarrow \frac{1}{5a + 4b + 3c + 12} \le \frac{1}{2a + b + 21} \text{ and analogously,}$$

$$\frac{1}{5b + 4c + 3a + 12} \le \frac{1}{2b + c + 21} \text{ and } \frac{1}{5c + 4a + 3b + 12} \le \frac{1}{2c + a + 21}$$

$$\therefore \frac{1}{5a + 4b + 3c + 12} + \frac{1}{5b + 4c + 3a + 12} + \frac{1}{5c + 4a + 3b + 12} \le \sum_{cyc} \frac{1}{2b + c + 21}$$

$$= \frac{1}{21}.\sum_{cyc} \frac{21 + 2b + c - (2b + c)}{2b + c + 21} = \frac{3}{21} - \frac{1}{21}.\sum_{cyc} \frac{2b + c}{2b + c + 21} \overset{?}{\le} \frac{1}{8}$$

$$\Leftrightarrow \frac{1}{21}.\sum_{cyc} \frac{2b + c}{2b + c + 21} \overset{?}{\ge} \frac{1}{56} \Leftrightarrow \sum_{cyc} \frac{2b + c}{2b + c + 21} \boxed{\overset{?}{\underset{(*)}{\ge}}} \frac{3}{8}$$

$$\text{Now, } \sum_{cyc} \frac{2b + c}{2b + c + 21} = 2.\sum_{cyc} \frac{b^2}{2b^2 + bc + 21b} + \sum_{cyc} \frac{c^2}{2bc + c^2 + 21c} \overset{Bergstrom}{\ge}$$

$$\frac{2\left(\sum_{cyc} a\right)^2}{2\sum_{cyc} a^2 + 4\sum_{cyc} ab - 3\sum_{cyc} ab + 21\sum_{cyc} a} + \frac{\left(\sum_{cyc} a\right)^2}{2\sum_{cyc} ab + \sum_{cyc} a^2 + 21\sum_{cyc} a}$$

$$\overset{AM-GM}{\ge} \frac{2\left(\sum_{cyc} a\right)^2}{2\left(\sum_{cyc} a\right)^2 - 9.\sqrt[3]{a^2b^2c^2} + 21\sum_{cyc} a} + \frac{\left(\sum_{cyc} a\right)^2}{\left(\sum_{cyc} a\right)^2 + 21\sum_{cyc} a} \overset{abc = 1}{=}$$

$$\frac{2t^2}{2t^2 - 9 + 21t} + \frac{t}{t + 21} \overset{?}{\ge} \frac{3}{8} \left(t = \sum_{cyc} a\right)$$

$$\Leftrightarrow 16t^2(t + 21) + 8t(2t^2 - 9 + 21t) \overset{?}{\ge} 3(t + 21)(2t^2 - 9 + 21t)$$

$$\Leftrightarrow 26t^3 + 315t^2 - 1368t + 567 \overset{?}{\ge} 0 \Leftrightarrow \left(\sum_{cyc} a - 3\right)(26t^2 + 393t - 189) \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because t = \sum_{cyc} a \overset{\text{via } ①}{\geq} 3 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{5a + 4b + 3c + 12} + \frac{1}{5b + 4c + 3a + 12} + \frac{1}{5c + 4a + 3b + 12} \leq \frac{1}{8}$$

$$\forall\, a, b, c > 0 \,\Big|\, ab + bc + ca = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$