ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$ and $xyz=1$ then prove that :

$$\frac{x}{\sqrt{y^2+2z}}+\frac{y}{\sqrt{z^2+2x}}+\frac{z}{\sqrt{x^2+2y}}\ge\sqrt{3}$$

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$$\frac{x}{\sqrt{y^2+2z}}+\frac{y}{\sqrt{z^2+2x}}+\frac{z}{\sqrt{x^2+2y}}=\sum_{cyc}\frac{x.\sqrt{x}}{\sqrt{xy^2+2zx}}\overset{Radon}{\ge}$$

$$\frac{\left(\sum_{cyc}x\right)^{\frac{3}{2}}}{\sqrt{\sum_{cyc}xy^2+2\sum_{cyc}xy}}\overset{?}{\ge}\sqrt{3}\Leftrightarrow\left(\sum_{cyc}x\right)^3\underset{(*)}{\underbrace{\overset{?}{\ge}}}3\sum_{cyc}xy^2+6\sum_{cyc}xy\ \ and$$

$$\because 1=\sqrt[3]{xyz}\overset{AM-GM}{\le}\frac{1}{3}\left(\sum_{cyc}x\right)\therefore 6\le 2\sum_{cyc}x\ \text{ and so, in order to prove } (*),$$

$$\text{it suffices to prove : }\left(\sum_{cyc}x\right)^3\overset{?}{\ge}3\sum_{cyc}xy^2+2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)$$

$$\Leftrightarrow\sum_{cyc}x^3+\sum_{cyc}x^2y\underset{(**)}{\underbrace{\overset{?}{\ge}}}2\sum_{cyc}xy^2$$

$$\text{Indeed, }x^3+z^2x\overset{AM-GM}{\ge}2zx^2,\ y^3+x^2y\overset{AM-GM}{\ge}2xy^2\ and\ z^3+y^2z\overset{AM-GM}{\ge}2yz^2$$

$$and\ \text{so, }\sum_{cyc}x^3+\sum_{cyc}x^2y\ge 2\sum_{cyc}xy^2\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{x}{\sqrt{y^2+2z}}+\frac{y}{\sqrt{z^2+2x}}+\frac{z}{\sqrt{x^2+2y}}\ge\sqrt{3}\ \forall\, x,y,z>0\,\middle|\,xyz=1,$$

$$''=''\ \text{ iff } x=y=z=1\ \text{(QED)}$$