ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $ab + bc + ca + abc = 4$ then prove that :

$$(a+1).\sqrt{(b+1)(c+1)} + (b+1).\sqrt{(c+1)(a+1)} + (c+1).\sqrt{(a+1)(b+1)} \geq a+b+c+9$$

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$$\sum_{cyc}\left((2+b)(2+c)\right) - (2+a)(2+b)(2+c) = 4 - (ab+bc+ca+abc)$$

$$= 0 \therefore \sum_{cyc}\frac{1}{2+a} = 1 \to (m) \ \& \ \frac{1}{2+a} \overset{a>0}{<} \frac{1}{2} \therefore \text{we can set} : \frac{1}{2+a} = \frac{1}{2} - x$$

$$\left(x > 0 \text{ and } x < \frac{1}{2}\right) \therefore a + 2 = \frac{2}{1-2x} \Rightarrow a = \frac{2}{1-2x} - 2 = \frac{2x}{\frac{1}{2} - x} \to (1)$$

$$\text{Similarly, we set} : \frac{1}{2+b} = \frac{1}{2} - y \text{ and } \frac{1}{2+c} = \frac{1}{2} - z \therefore 1 \overset{\text{via }(m)}{=}$$

$$\frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} = \frac{1}{2} - x + \frac{1}{2} - y + \frac{1}{2} - z \Rightarrow x + y + z = \frac{1}{2} \to (i)$$

$$\therefore (1) \text{ and } (i) \Rightarrow a = \frac{2x}{y+z} \text{ and analogously, } b = \frac{2y}{z+x} \text{ and } c = \frac{2z}{x+y} \text{ and hence :}$$

$$\text{LHS} - \text{RHS} = \sum_{cyc}\left(\left(\frac{2x}{y+z} + 1\right).\sqrt{\left(\frac{2y}{z+x} + 1\right)\left(\frac{2z}{x+y} + 1\right)}\right) - \sum_{cyc}\left(\frac{2x}{y+z} + 2\right) - 3$$

$$= \sum_{cyc}\left(\frac{B+C}{A}.\sqrt{\frac{(C+A)(A+B)}{BC}}\right) - 2\left(\sum_{cyc}x\right)\left(\sum_{cyc}\frac{1}{A}\right) - 3$$

$$(y+z = A, z+x = B, x+y = C) \overset{?}{\geq} 0$$

$$\Leftrightarrow \sqrt{\frac{(B+C)(C+A)(A+B)}{ABC}}.\sum_{cyc}\sqrt{\frac{B+C}{A}} \overset{?}{\geq} 2\left(\sum_{cyc}x\right)\left(\sum_{cyc}\frac{1}{A}\right) + 3$$

$$\Leftrightarrow \frac{(B+C)(C+A)(A+B)}{ABC}.\left(\sum_{cyc}\frac{B+C}{A} + 2\sum_{cyc}\sqrt{\frac{(B+C)(C+A)}{AB}}\right)$$

$$\overset{?}{\underset{(*)}{\geq}} \left(2\left(\sum_{cyc}x\right)\left(\sum_{cyc}\frac{1}{A}\right) + 3\right)^2 \ \& \ A + B - C = 2z > 0, B + C - A = 2x > 0 \text{ and}$$

$C + A - B = 2y > 0 \Rightarrow A + B > C, B + C > A, C + A > B \Rightarrow A, B, C$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) and then : $2\sum_{cyc} x = \sum_{cyc} A = 2s \Rightarrow \sum_{cyc} x = s$ and then, LHS of $(*)$ − RHS of $(*)$ =

$$\frac{2s(s^2+2Rr+r^2)}{4Rrs}\cdot\left(\frac{2s(s^2+4Rr+r^2)-12Rrs}{4Rrs}+2\sum_{cyc}\sqrt{\frac{(B+C)(C+A)}{AB}}\right)-$$

$$\left(\frac{2s(s^2+4Rr+r^2)+12Rrs}{4Rrs}\right)^2 \overset{AM-GM}{\geq}$$

$$\frac{s^2+2Rr+r^2}{2Rr}\cdot\left(\frac{s^2-2Rr+r^2}{2Rr}+6.\sqrt[3]{\frac{(B+C)(C+A)(A+B)}{ABC}}\right)-$$

$$\left(\frac{s^2+10Rr+r^2}{2Rr}\right)^2 \overset{Cesaro}{\geq} \frac{s^2+2Rr+r^2}{2Rr}\cdot\left(\frac{s^2-2Rr+r^2}{2Rr}+12\right)-$$

$$\left(\frac{s^2+10Rr+r^2}{2Rr}\right)^2 = \frac{(s^2+2Rr+r^2)(s^2+22Rr+r^2)-(s^2+10Rr+r^2)^2}{4R^2r^2s^2}$$

$$= \frac{4Rr(s^2-14Rr+r^2)}{4R^2r^2s^2} = \frac{4Rr\left((s^2-16Rr+5r^2)+2r(R-2r)\right)}{4R^2r^2s^2} \overset{\substack{\text{Gerretsen} \\ \text{and} \\ \text{Euler}}}{\geq} 0 \Rightarrow$$

$(*)$ is true $\therefore (a+1).\sqrt{(b+1)(c+1)}+(b+1).\sqrt{(c+1)(a+1)}+$
$(c+1).\sqrt{(a+1)(b+1)} \geq a+b+c+9\ \forall\, a,b,c>0 \mid ab+bc+ca+abc=4$,
$''=''$ iff $a=b=c=1$ (QED)