ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ then prove that :

$$(x + y - z)(y + z - x)(z + x - y) \le 1$$

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$$\boxed{\text{Case 1}}\ xyz \le 1 \text{ and } (x + y - z)(y + z - x)(z + x - y) \le 1$$

$$\Leftrightarrow 1 + \sum_{cyc} x^3 + 5xyz \ge \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \text{ and } \because 1 \ge xyz$$

$$\therefore \text{ it suffices to prove : } \sum_{cyc} x^3 + 6xyz \ge \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \to \text{true via Schur}$$

$$\boxed{\text{Case 2}}\ xyz \ge 1 \text{ and then : } \frac{xyz(\sum_{cyc} x)}{\sum_{cyc} xy} \cdot \sqrt{\frac{xyz(\sum_{cyc} x)}{\sum_{cyc} xy}} \le xyz \left(\because 1 = \frac{xyz(\sum_{cyc} x)}{\sum_{cyc} xy}\right)$$

$$\Rightarrow xyz\left(\sum_{cyc} x\right)^3 \le \left(\sum_{cyc} xy\right)^3 \to (1)\ \&\ \text{assigning } y + z = a, z + x = b, x + y = c \Rightarrow$$

$$a + b - c = 2z > 0, b + c - a = 2x > 0 \text{ and } c + a - b = 2y > 0 \Rightarrow a + b > c,$$

$$b + c > a, c + a > b\ \Rightarrow a, b, c \text{ form sides of a triangle with semiperimeter,}$$

$$\text{circumradius and inradius} = s, R, r \text{ (say)} \Rightarrow \sum_{cyc} x = s, xyz = r^2 s,$$

$$\sum_{cyc} xy = 4Rr + r^2, \sum_{cyc} x^3 = s^3 - 12Rrs \text{ and then : } (1) \Rightarrow r^2 s^4 \le (4Rr + r^2)^3 \Rightarrow$$

$$r \ge \frac{s^4}{(4R + r)^3} \Rightarrow \frac{xyz(\sum_{cyc} x)}{\sum_{cyc} xy} = \frac{r^2 s^2}{4Rr + r^2} = \frac{rs^2}{4R + r} \ge \frac{s^4}{(4R + r)^3} \cdot \frac{s^2}{4R + r}$$

$$= \frac{s^6}{(4R + r)^4} \Rightarrow \frac{xyz(\sum_{cyc} x)}{\sum_{cyc} xy} \ge \frac{s^6}{(4R + r)^4} \therefore 1 = \frac{xyz(\sum_{cyc} x)}{\sum_{cyc} xy} \cdot \sqrt{\frac{xyz(\sum_{cyc} x)}{\sum_{cyc} xy}} \ge$$

$$\frac{rs^2}{4R + r} \cdot \sqrt{\frac{s^6}{(4R + r)^4}} = \frac{rs^5}{(4R + r)^3} \text{ and so, } 1 + \sum_{cyc} x^3 + 5xyz - \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)$$

$$\ge \frac{rs^5}{(4R + r)^3} + s^3 - 12Rrs + 5r^2 s - s(4Rr + r^2) \overset{?}{\ge} 0$$

$$\Leftrightarrow rs^4 + (s^2 - 16Rr + 4r^2)(4R + r)^3 \boxed{\overset{?}{\underset{(*)}{\geq}}} 0 \text{ and } \because r(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} r(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (64R^3 + 48R^2r + 44Rr^2 - 9r^3)s^2 \boxed{\overset{?}{\underset{(**)}{\geq}}} r\begin{pmatrix} 1024R^4 + 512R^3r + \\ 256R^2r^2 - 192Rr^3 + 21r^4 \end{pmatrix}$$

$$\text{Now, } (R - r)(s^2 - 16Rr + 5r^2) \overset{\text{Rouche}}{\geq}$$

$$(R - r)\left(2R^2 - 6Rr + 4r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr}\right)$$

$$= (R - 2r)\left(\left(R - r - \sqrt{R^2 - 2Rr}\right)^2 + r^2\right) \overset{\text{Euler}}{\geq} r^2(R - 2r) \Rightarrow$$

$$(R - r)(s^2 - 16Rr + 5r^2) \geq r^2(R - 2r) \Rightarrow s^2 \geq 16Rr - 5r^2 + \frac{r^2(R - 2r)}{R - r} \text{ and so,}$$

$$(64R^3 + 48R^2r + 44Rr^2 - 9r^3)s^2 \geq$$

$$(64R^3 + 48R^2r + 44Rr^2 - 9r^3)\left(16Rr - 5r^2 + \frac{r^2(R - 2r)}{R - r}\right) \overset{?}{\geq} \text{RHS of } (**)$$

$$\Leftrightarrow 64t^3 - 144t^2 + 33t - 2 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t - 2)(8t - 1)^2 \overset{?}{\geq} 0 \rightarrow \text{true via Euler}$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \text{combining both cases}, \therefore (x + y - z)(y + z - x)(z + x - y)$$

$$\leq 1 \; \forall\, x, y, z > 0 \,\Big|\, x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, " = " \text{ iff } x = y = z = 1 \text{ (QED)}$$