ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $ab + bc + ca = abc$ then prove that :

$$(a + b - c - 1)(b + c - a - 1)(c + a - b - 1) \le 8$$

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$$(a + b - c - 1)(b + c - a - 1)(c + a - b - 1) \le 8$$

$$\Leftrightarrow \sum_{cyc} a^3 + 5abc + 2\sum_{cyc} ab + 9 \ge \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) + \sum_{cyc} a^2 + \sum_{cyc} a$$

$$\Leftrightarrow \sum_{cyc} a^3 + 7abc + 9\left(\frac{abc}{\sum_{cyc} ab}\right)^3 \ge$$

$$\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) + \left(\sum_{cyc} a^2\right)\left(\frac{abc}{\sum_{cyc} ab}\right) + \left(\sum_{cyc} a\right)\left(\frac{abc}{\sum_{cyc} ab}\right)^2 \Leftrightarrow$$

$$\left(\sum_{cyc} a^3 + 7abc - \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)\right)\left(\sum_{cyc} ab\right)^3 + 9a^3b^3c^3 \boxed{\overset{(*)}{\ge}}$$

$$abc\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)^2 + a^2b^2c^2\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); then : $\sum_{cyc} a = s \to (1), abc = r^2 s \to (2), \sum_{cyc} ab = 4Rr + r^2 \to (3),$

$$\sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4), \sum_{cyc} a^3 = s^3 - 12Rrs \to (5) \therefore (*)$$

$$\overset{\text{via (1),(2),(3),(4) and (5)}}{\Leftrightarrow} \left(s^3 - 12Rrs + 7r^2 s - s(4Rr + r^2)\right)(4Rr + r^2)^3 + 9r^6 s^3 \ge$$

$$r^2 s(s^2 - 8Rr - 2r^2)(4Rr + r^2)^2 + r^4 s^3(4Rr + r^2)$$

$$\Leftrightarrow (8R^3 + 4R^2 r + r^3)s^2 \boxed{\underset{(**)}{\overset{?}{\ge}}} r(128R^4 + 32R^3 r - 24R^2 r^2 - 10Rr^3 - r^4)$$

$$\text{Now, } (R - r)(s^2 - 16Rr + 5r^2) \overset{\text{Rouche}}{\ge}$$

$$(R - r)\left(2R^2 - 6Rr + 4r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr}\right)$$

$$= (R - 2r)\left(\left(R - r - \sqrt{R^2 - 2Rr}\right)^2 + r^2\right) \overset{\text{Euler}}{\ge} r^2(R - 2r) \Rightarrow$$

$$(\mathbf{R}-\mathbf{r})(\mathbf{s}^2-16\mathbf{Rr}+5\mathbf{r}^2)\geq\mathbf{r}^2(\mathbf{R}-2\mathbf{r})\Rightarrow\mathbf{s}^2\geq16\mathbf{Rr}-5\mathbf{r}^2+\frac{\mathbf{r}^2(\mathbf{R}-2\mathbf{r})}{\mathbf{R}-\mathbf{r}}\ \boldsymbol{and\ so},$$

$$(8\mathbf{R}^3+4\mathbf{R}^2\mathbf{r}+\mathbf{r}^3)\mathbf{s}^2\geq(8\mathbf{R}^3+4\mathbf{R}^2\mathbf{r}+\mathbf{r}^3)\left(16\mathbf{Rr}-5\mathbf{r}^2+\frac{\mathbf{r}^2(\mathbf{R}-2\mathbf{r})}{\mathbf{R}-\mathbf{r}}\right)$$

$$\overset{?}{\geq}\mathbf{r}(128\mathbf{R}^4+32\mathbf{R}^3\mathbf{r}-24\mathbf{R}^2\mathbf{r}^2-10\mathbf{Rr}^3-\mathbf{r}^4)\Leftrightarrow8\mathbf{r}^7\mathbf{s}(\mathbf{R}-2\mathbf{r})(14\mathbf{R}-\mathbf{r})\overset{?}{\geq}0$$

$$\rightarrow\textbf{true via Euler}\Rightarrow(**)\Rightarrow(*)\ \textbf{is true}$$

$$\therefore(\boldsymbol{a}+\mathbf{b}-\mathbf{c}-1)(\mathbf{b}+\mathbf{c}-\boldsymbol{a}-1)(\mathbf{c}+\boldsymbol{a}-\mathbf{b}-1)\leq8$$

$$\forall\,\boldsymbol{a},\mathbf{b},\mathbf{c}>0\,\Big|\ \boldsymbol{a}\mathbf{b}+\mathbf{bc}+\mathbf{c}\boldsymbol{a}=\boldsymbol{a}\mathbf{bc},''=''\ \textbf{iff}\ \boldsymbol{a}=\mathbf{b}=\mathbf{c}=3\ (\mathbf{QED})$$