ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a+b+c=3$ then prove that :

$$2\sqrt{a^2b+bc^2+ca^2}+2\sqrt{ab^2+bc^2+ca^2}\ge 3\sqrt{ab+bc+ca}+\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Upon squaring, desired inequality becomes :

$$4\sum_{cyc}a^2b+4\sum_{cyc}ab^2+8.\sqrt{\left(\sum_{cyc}a^2b\right)\left(\sum_{cyc}ab^2\right)}\overset{(*)}{\ge}$$

$$9\sum_{cyc}ab+\sum_{cyc}a^2b^2+6.\sqrt{\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b^2\right)}$$

$$\text{Now, }\left(\sum_{cyc}a^2b\right)\left(\sum_{cyc}ab^2\right)-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b^2\right)=$$

$$abc\sum_{cyc}a^3+\sum_{cyc}a^3b^3+3a^2b^2c^2-\left(\sum_{cyc}a^3b^3+abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\right)$$

$$=abc\left(\sum_{cyc}a^3+3abc-\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\right)\overset{\text{Schur}}{\ge}0\ \therefore\ 6.\sqrt{\left(\sum_{cyc}a^2b\right)\left(\sum_{cyc}ab^2\right)}$$

$$\ge 6.\sqrt{\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b^2\right)}\ \therefore\ \text{LHS of }(*)-\text{RHS of }(*)\ge$$

$$4\sum_{cyc}a^2b+4\sum_{cyc}ab^2+2.\sqrt{\left(\sum_{cyc}a^2b\right)\left(\sum_{cyc}ab^2\right)}-9\sum_{cyc}ab-\sum_{cyc}a^2b^2\overset{\text{AM-GM}}{\ge}$$

$$4\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-12abc+6abc-9\sum_{cyc}ab-\sum_{cyc}a^2b^2$$

$$\overset{a+b+c=3}{=}4\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-6abc-3\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-\frac{3\sum_{cyc}a^2b^2}{\sum_{cyc}a}$$

$$=\frac{\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}ab\right)-6abc\left(\sum_{cyc}a\right)-3\sum_{cyc}a^2b^2}{\sum_{cyc}a}=$$

$$\frac{\sum_{cyc}a^3b+\sum_{cyc}ab^3-\sum_{cyc}a^2b^2-abc\left(\sum_{cyc}a\right)}{\sum_{cyc}a}\overset{AM-GM}{\geq}\frac{2\sum_{cyc}a^2b^2-\sum_{cyc}a^2b^2-abc\left(\sum_{cyc}a\right)}{\sum_{cyc}a}$$

$$=\frac{\sum_{cyc}(a^2(b-c)^2)}{2\sum_{cyc}a}\geq 0\Rightarrow(*)\text{ is true}\therefore 2\sqrt{a^2b+bc^2+ca^2}+2\sqrt{ab^2+bc^2+ca^2}$$

$$\geq 3\sqrt{ab+bc+ca}+\sqrt{a^2b^2+b^2c^2+c^2a^2}\ \forall\, a,b,c>0\,\big|\,a+b+c=3,$$

$$''='' \text{ iff } a=b=c=1\ (\text{QED})$$