ROMANIAN MATHEMATICAL MAGAZINE

If $0 < x, y, z \leq 1$ then prove that :

$$\frac{xy+1}{x+y} + \frac{yz+1}{y+z} + \frac{zx+1}{z+x} \geq \frac{xy+yz+zx+3}{x+y+z} + 1$$

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$$0 < x, y \leq 1 \Rightarrow (x-1), (y-1) \leq 0 \Rightarrow (x-1)(y-1) \geq 0$$

$$\Rightarrow xy + 1 \geq x + y \text{ and analogously}, yz + 1 \geq y + z \text{ and } zx + 1 \geq z + x \rightarrow (1)$$

$$\text{and } \frac{xy+1}{x+y} + \frac{yz+1}{y+z} + \frac{zx+1}{z+x} \geq \frac{xy+yz+zx+3}{x+y+z} + 1$$

$$\Leftrightarrow \sum_{cyc} \frac{xy+1}{x+y} \geq \sum_{cyc} \frac{xy+1}{x+y+z} + 1 \Leftrightarrow \sum_{cyc} \left((xy+1) \left(\frac{1}{x+y} - \frac{1}{x+y+z} \right) \right) \geq 1$$

$$\Leftrightarrow \sum_{cyc} \left(\frac{z(xy+1)}{(x+y)(x+y+z)} \right) \overset{(*)}{\geq} 1$$

$$\text{Now, via } (1), \text{LHS of } (*) \geq \sum_{cyc} \left(\frac{z(x+y)}{(x+y)(x+y+z)} \right) = \frac{1}{x+y+z} \cdot \sum_{cyc} x = 1$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{xy+1}{x+y} + \frac{yz+1}{y+z} + \frac{zx+1}{z+x} \geq \frac{xy+yz+zx+3}{x+y+z} + 1$$

$$\forall\, x, y, z > 0 \,\big|\, 0 < x, y, z \leq 1, '' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$