ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x + y + z = 3$ then prove that :

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}+\frac{2}{21-9xyz}\ge\frac{5}{3}$$

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$$\text{Assigning } y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0,$$
$$b+c-a=2x>0 \text{ and } c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b$$
$$\Rightarrow a,b,c \text{ form sides of a triangle with semiperimeter, circumradius and inradius}$$
$$= s, R, r \text{ (say)} \Rightarrow \sum_{\text{cyc}} x = s, xyz = r^2 s \text{ and so,}$$

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}+\frac{2}{21-9xyz}\overset{?}{\ge}\frac{5}{3}\overset{x+y+z=3}{\Leftrightarrow}$$

$$\sum_{\text{cyc}}\frac{1}{a}+\frac{\frac{2}{9}\left(\sum_{\text{cyc}}x\right)^2}{\frac{21}{27}\left(\sum_{\text{cyc}}x\right)^3-9xyz}\overset{?}{\ge}\frac{5}{\sum_{\text{cyc}}x}\Leftrightarrow\frac{s^2+4Rr+r^2}{4Rrs}\overset{?}{\ge}\frac{5}{3}-\frac{2s^2}{7s^3-81r^2s}$$

$$\Leftrightarrow (s^2+4Rr+r^2)(7s^2-81r^2)\overset{?}{\ge}4Rr(33s^2-405r^2)$$

$$\Leftrightarrow 7s^4-(104Rr+74r^2)+r^3(1296R-81r)\boxed{\underset{(*)}{\overset{?}{\ge}}}0 \text{ and } \because P =$$

$$7(s^2-16Rr+5r^2)^2+8r(15R-18r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0 \therefore \text{in order}$$

$$\text{to prove } (*), \text{it suffices to prove : LHS of } (*)\overset{?}{\ge}P\Leftrightarrow 8r^2(16R-29r)(R-2r)\overset{?}{\ge}0$$

$$\rightarrow \text{true} \because R\overset{\text{Euler}}{\ge}2r\Rightarrow(*) \text{ is true} \therefore \frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}+\frac{2}{21-9xyz}\ge\frac{5}{3}$$

$$\forall x,y,z>0 \,\middle|\, x+y+z=3, \text{"="} \text{ iff } x=y=z=1 \text{ (QED)}$$