ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a^3 + b^3 + c^3 = a + b + c$ then prove that :

$$\frac{a^3}{a^4 + b + c} + \frac{b^3}{b^4 + c + a} + \frac{c^3}{c^4 + a + b} \leq 1$$

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$$\frac{a^4 + b + c}{a + b + c} = \frac{a^3 . a + 1 . b + 1 . c}{a + b + c} \overset{\text{Weighted AM}-\text{Weighted HM}}{\geq} \frac{a + b + c}{\frac{a}{a^3} + \frac{b}{1} + \frac{c}{1}}$$

$$= \frac{a^2(a + b + c)}{1 + a^2(b + c)} \Rightarrow a^4 + b + c \geq \frac{a^2(a + b + c)^2}{1 + a^2(b + c)} \Rightarrow \frac{a^3}{a^4 + b + c} \leq \frac{a\left(1 + a^2(b + c)\right)}{(a + b + c)^2}$$

$$\text{and analogs} \Rightarrow \sum_{\text{cyc}} \frac{a^3}{a^4 + b + c} \leq \sum_{\text{cyc}} \frac{a\left(1 + a^2(b + c)\right)}{(a + b + c)^2} \overset{?}{\leq} 1$$

$$\Leftrightarrow \sum_{\text{cyc}} a + \sum_{\text{cyc}} \left(a^3 \left(\sum_{\text{cyc}} a - a\right)\right) \overset{?}{\leq} \left(\sum_{\text{cyc}} a\right)^2$$

$$\Leftrightarrow \left(\sum_{\text{cyc}} a\right)^2 + \sum_{\text{cyc}} a^4 \overset{?}{\geq} \sum_{\text{cyc}} a + \left(\sum_{\text{cyc}} a\right)\left(\sum_{\text{cyc}} a^3\right)$$

$$\overset{a^3+b^3+c^3 = a+b+c}{\Leftrightarrow} \left(\sum_{\text{cyc}} a\right)^2 + \sum_{\text{cyc}} a^4 \overset{?}{\geq} \sum_{\text{cyc}} a + \left(\sum_{\text{cyc}} a\right)^2 \overset{a^3+b^3+c^3 = a+b+c}{\Leftrightarrow}$$

$$\sum_{\text{cyc}} a^4 \overset{?}{\geq} \left(\sum_{\text{cyc}} a\right)\left(\frac{\sum_{\text{cyc}} a^3}{\sum_{\text{cyc}} a}\right)^{\frac{3}{2}} \Leftrightarrow \left(\sum_{\text{cyc}} a^4\right)^2 \overset{?}{\geq} \left(\sum_{\text{cyc}} a\right)^2 . \left(\frac{\sum_{\text{cyc}} a^3}{\sum_{\text{cyc}} a}\right)^3$$

$$\Leftrightarrow \left(\sum_{\text{cyc}} a^4\right)^2 . \left(\sum_{\text{cyc}} a\right) \overset{?}{\geq} \left(\sum_{\text{cyc}} a^3\right)^3 \rightarrow \text{true} \because \left(\sum_{\text{cyc}} a^4\right)^2 . \left(\sum_{\text{cyc}} a\right)$$

$$= \left(\sum_{\text{cyc}} a^4\right) . \left(\sum_{\text{cyc}} a^4\right) . \left(\sum_{\text{cyc}} a\right) \overset{\text{Holder}}{\geq} \left(\sum_{\text{cyc}} a^3\right)^3$$

$$\therefore \frac{a^3}{a^4 + b + c} + \frac{b^3}{b^4 + c + a} + \frac{c^3}{c^4 + a + b} \leq 1$$

$$\forall\, a, b, c > 0 \,\middle|\, a^3 + b^3 + c^3 = a + b + c, \text{"="} \text{ iff } a = b = c = 1 \text{ (QED)}$$