ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$ then:

$$(x^2-xy+y^2)(y^2-yz+z^2)(z^2-zx+x^2)\geq\frac{1}{3}xyz(x^3+y^3+z^3)$$

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$$\textit{We need to show}:$$

$$(x^2-xy+y^2)(y^2-yz+z^2)(z^2-zx+x^2)\geq\frac{1}{3}xyz(x^3+y^3+z^3)$$

$$3(x^2-xy+y^2)(y^2-yz+z^2)(z^2-zx+x^2)\geq xyz(x^3+y^3+z^3)$$

$$3(x+y)(x^2-xy+y^2)(y+z)(y^2-yz+z^2)(z+x)(z^2-zx+x^2)$$
$$\geq xyz(x^3+y^3+z^3)\prod(x+y)$$

(Multiplying both sides by (x+y)(y+z)(z+x))

$$LHS=3(x+y)(x^2-xy+y^2)(y+z)(y^2-yz+z^2)(z+x)(z^2-zx+x^2)=$$

$$=3(x^3+y^3)(y^3+z^3)(z^3+x^3)=3\left(\sum x^3\sum x^3y^3-x^3y^3z^3\right)=$$
$$=3\left(\sum x^3\right)\left(\sum x^3y^3\right)-3x^3y^3z^3=$$
$$=3\left(\sum x^3\right)\left(\left(\sum xy\right)^3-3(xy+yz)(yz+zx)(zx+xy)\right)-3x^3y^3z^3=$$
$$=\left(\sum x^3\right)\left(3\left(\sum xy\right)^3-9xyz(x+y)(y+z)(z+x)\right)-3x^3y^3z^3(1)$$

We need to show:

$$3(x+y)(x^2-xy+y^2)(y+z)(y^2-yz+z^2)(z+x)(z^2-zx+x^2)\geq$$
$$\geq xyz(x^3+y^3+z^3)\prod(x+y)$$

$$\left(\sum x^3\right)\left(3\left(\sum xy\right)^3-9xyz(x+y)(y+z)(z+x)\right)-3x^3y^3z^3\geq$$
$$\geq xyz(x^3+y^3+z^3)\prod(x+y)$$

(using relation (1))

$$\left(\sum x^3\right)\left(3\left(\sum xy\right)^3-10xyz(x+y)(y+z)(z+x)\right)\geq 3x^3y^3z^3$$

$$\left(\sum x^3\right)\left(3\left(\sum xy\right)^3-10xyz(x+y)(y+z)(z+x)\right)\geq 3x^3y^3z^3$$

$$\textit{This is true because}:$$

$$\left(\sum x^3\right)\left(3\left(\sum xy\right)^3-10xyz(x+y)(y+z)(z+x)\right)=$$

$$=\left(\sum x^3\right)\left(3\left(\sum xy\right)^3-10(xy+yz)(yz+zx)(zx+xy)\right)\overset{AM-GM}{\geq}$$

$$\geq\left(\sum x^3\right)\left(3\left(\sum xy\right)^3-\frac{10(2xy+2yz+2zx)^3}{27}\right)$$

$$=\sum x^3\left(3\left(\sum xy\right)^3-\frac{80}{27}\left(\sum xy\right)^3\right)$$

$$=\frac{(\sum x^3)(\sum xy)^3}{27}\overset{AM-GM}{\geq}3xyz.\frac{27(xyz)^2}{27}=3x^3y^3z^3$$

Equality holds for x=y=z.