ROMANIAN MATHEMATICAL MAGAZINE

If $a,\mathrm{b}>0$ and $a^3-ab+\mathrm{b}^3\le 1$ then prove that :

$$a^2+\mathrm{b}^2\le 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\boxed{\text{Case 1}}\ ab\ge 1 \text{ and then : } a^3-ab+\mathrm{b}^3\le 1\Rightarrow 1+ab\ge a^3+\mathrm{b}^3$$
$$\ge\frac{1}{4}(a+\mathrm{b})^3\Rightarrow(a+\mathrm{b})^2\le\sqrt[3]{16(1+ab)^2}\Rightarrow(a+\mathrm{b})^2-2ab\le$$
$$\sqrt[3]{16(1+ab)^2}-2ab\overset{?}{\le}2\Leftrightarrow 16(1+ab)^2\overset{?}{\le}8(1+ab)^3\Leftrightarrow 1+ab\overset{?}{\ge}2\Leftrightarrow ab\overset{?}{\ge}1$$
$$\to\text{true}\ \therefore a^2+\mathrm{b}^2\le 2$$
$$\boxed{\text{Case 2}}\ ab\le 1 \text{ and then : } a^3-ab+\mathrm{b}^3\le 1\Rightarrow a^3+\mathrm{b}^3\le 1+ab\le 2\to (1)$$
$$\text{Now, via Power}-\text{Mean Inequality, }\left(\frac{a^3+\mathrm{b}^3}{2}\right)^{\frac{1}{3}}\ge\left(\frac{a^2+\mathrm{b}^2}{2}\right)^{\frac{1}{2}}$$
$$\Rightarrow\sqrt{\frac{a^2+\mathrm{b}^2}{2}}\le\left(\frac{a^3+\mathrm{b}^3}{2}\right)^{\frac{1}{3}}\overset{\text{via }(1)}{\le}1\Rightarrow a^2+\mathrm{b}^2\le 2\ \therefore\text{ combining both cases,}$$
$$a^2+\mathrm{b}^2\le 2\ \forall\, a,\mathrm{b}>0\ \middle|\ a^3-ab+\mathrm{b}^3\le 1,''=''\text{ iff } a=\mathrm{b}=1\ (\text{QED})$$