

ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0, x + y \leq 2$ then:

$$\frac{x^2}{x^2 + y} + \frac{y^2}{y^2 + x} \leq 1$$

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$$\begin{aligned} \frac{x^2}{x^2 + y} + \frac{y^2}{y^2 + x} &= \sum \frac{x^2}{x^2 + y} = \sum \left(1 - \frac{y}{x^2 + y} \right) = 2 - \sum \frac{y}{x^2 + y} = \\ &= 2 - \sum \frac{y^2}{x^2 y + y^2} \stackrel{\text{Bergstrom}}{\leq} 2 - \frac{(x + y)^2}{xy(x + y) + x^2 + y^2} = \\ &= 2 - \frac{(x + y)^2}{xy(x + y) + (x + y)^2 - 2xy} \stackrel{x+y \leq 2}{\leq} 2 - \frac{(x + y)^2}{xy(2) + (x + y)^2 - 2xy} = \\ &= 2 - \frac{(x + y)^2}{(x + y)^2} = 2 - 1 = 1 \end{aligned}$$

Equality holds for $x = y = 1$.