ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, then prove that :

$$\frac{4(x^2+y^2+z^2)}{27(xy+yz+zx)}+\frac{x}{7x+y+z}+\frac{y}{x+7y+z}+\frac{z}{x+y+7z}\ge\frac{13}{27}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{4(x^2+y^2+z^2)}{27(xy+yz+zx)}+\frac{x}{7x+y+z}+\frac{y}{x+7y+z}+\frac{z}{x+y+7z}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{4\sum_{cyc}x^2}{27\sum_{cyc}xy}+\frac{\left(\sum_{cyc}x\right)^2}{7\sum_{cyc}x^2+2\sum_{cyc}xy}=\frac{4m}{27n}+\frac{m+2n}{7m+2n}\left(m=\sum_{cyc}x^2,n=\sum_{cyc}xy\right)$$

$$=\frac{4m(7m+2n)+27n(m+2n)}{27n(7m+2n)}\overset{?}{\ge}\frac{13}{27}\Leftrightarrow 28(m-n)^2\overset{?}{\ge}0\to\text{true}$$

$$\therefore\frac{4(x^2+y^2+z^2)}{27(xy+yz+zx)}+\frac{x}{7x+y+z}+\frac{y}{x+7y+z}+\frac{z}{x+y+7z}\ge\frac{13}{27}$$

$$\forall\, x,y,z>0,''=''\ \text{iff}\ x=y=z\ (\text{QED})$$