ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0,\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=3$ and $\lambda\geq 0$ then :

$$\sum_{cyc}\frac{1}{a+\lambda\sqrt{bc}+1}\leq\frac{3}{\lambda+2}$$

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$$\text{Let }\sqrt{a}=x,\sqrt{b}=y,\sqrt{c}=z\text{ and then }:\sum_{cyc}\frac{1}{a+\lambda.\sqrt{bc}+1}\leq\frac{3}{\lambda+2}\Leftrightarrow$$

$$\lambda^3\left(3x^2y^2z^2-xyz\sum_{cyc}x\right)+\lambda^2\left(\begin{matrix}3xyz\sum_{cyc}x^3-\sum_{cyc}x^3y-\sum_{cyc}xy^3+\\ xyz\sum_{cyc}x-2\sum_{cyc}xy\end{matrix}\right)+$$

$$\lambda\left(3\sum_{cyc}x^3y^3+\sum_{cyc}x^3y+\sum_{cyc}xy^3-\sum_{cyc}x^2y^2-2\sum_{cyc}x^2-\sum_{cyc}xy-3\right)+3x^2y^2z^2$$

$$-3+\sum_{cyc}x^2y^2-\sum_{cyc}x^2\ \boxed{\overset{(*)}{\geq}}\ 0\text{ and},3=\sum_{cyc}\frac{1}{x^4}\overset{\text{Radon}}{\geq}\frac{3^5}{\left(\sum_{cyc}x\right)^4}\Rightarrow\sum_{cyc}x\geq 3\rightarrow (1)$$

$$\text{and also},3=\sum_{cyc}\frac{1}{x^4}\overset{\text{AM-GM}}{\geq}\frac{3}{\sqrt[3]{x^4y^4z^4}}\Rightarrow xyz\geq 1\rightarrow (2)\ \&\ \text{now},3xyz\overset{?}{\geq}\sum_{cyc}x\Leftrightarrow$$

$$9x^2y^2z^2\overset{?}{\geq}\left(\sum_{cyc}x\right)^2\cdot\frac{3x^4y^4z^4}{\sum_{cyc}x^4y^4}\left(\because 3=\sum_{cyc}\frac{1}{x^4}\right)\Leftrightarrow 3\sum_{cyc}x^4y^4\overset{?}{\geq}x^2y^2z^2\left(\sum_{cyc}x\right)^2$$

$$\rightarrow\text{true}\because 3\sum_{cyc}x^4y^4\geq 3x^2y^2z^2\sum_{cyc}x^2\geq x^2y^2z^2\left(\sum_{cyc}x\right)^2\ \therefore 3xyz\ \boxed{\boxed{\overset{(\bullet)}{\geq}}}\ \sum_{cyc}x$$

$$\Rightarrow xyz\sum_{cyc}x\geq\frac{1}{3}\left(\sum_{cyc}x\right)^2\geq\sum_{cyc}xy\ \therefore xyz\sum_{cyc}x\ \boxed{\boxed{\overset{(\bullet\bullet)}{\geq}}}\ \sum_{cyc}xy\text{ and}\left(\sum_{cyc}x^2y^2\right)^2\overset{?}{\geq}$$

$$\left(\sum_{cyc}x^2\right)^2\cdot\frac{3x^4y^4z^4}{\sum_{cyc}x^4y^4}\Leftrightarrow\left(\sum_{cyc}x^4y^4\right)\left(\sum_{cyc}x^2y^2\right)^2\overset{?}{\geq}3x^4y^4z^4\left(\sum_{cyc}x^2\right)^2\rightarrow\text{true}$$

$$\because\left(\sum_{cyc}x^4y^4\right)\left(\sum_{cyc}x^2y^2\right)^2\geq x^2y^2z^2\left(\sum_{cyc}x^2\right)\left(3x^2y^2z^2\left(\sum_{cyc}x^2\right)\right)$$

$$\therefore \sum_{cyc} x^2y^2 \boxed{\overset{(\bullet\bullet\bullet)}{\geq}} \sum_{cyc} x^2 \text{ and } 3xyz\sum_{cyc} x^3 - \sum_{cyc} x^3y - \sum_{cyc} xy^3 + xyz\sum_{cyc} x - 2\sum_{cyc} xy$$

$$\overset{\text{via ② and } (\bullet\bullet)}{\geq} 2xyz\sum_{cyc} x^3 + \sum_{cyc} x^3 - \sum_{cyc} xy - \sum_{cyc} x^3y - \sum_{cyc} xy^3 \overset{\text{via Chebyshev and ①}}{\geq}$$

$$2xyz\sum_{cyc} x^3 + \sum_{cyc} x^2 - \sum_{cyc} xy - \sum_{cyc} x^3y - \sum_{cyc} xy^3 \geq 2xyz\sum_{cyc} x^3 - \sum_{cyc} x^3y$$

$$-\sum_{cyc} xy^3 \overset{?}{\geq} 0 \Leftrightarrow 4x^2y^2z^2\left(\sum_{cyc} x^3\right)^2 \overset{?}{\geq} \left(\sum_{cyc} x^3y + \sum_{cyc} xy^3\right)^2 \cdot \frac{3x^4y^4z^4}{\sum_{cyc} x^4y^4}$$

$$\Leftrightarrow 4\left(\sum_{cyc} x^4y^4\right)\left(\sum_{cyc} x^3\right)^2 \overset{?}{\geq} 3x^2y^2z^2\left(\sum_{cyc} x^3y + \sum_{cyc} xy^3\right)^2 \to \text{true} \because \text{LHS} \geq$$

$$\frac{4}{3}x^2y^2z^2\left(\sum_{cyc} x\right)^2\left(\sum_{cyc} x^3\right)^2 \overset{?}{\geq} \text{RHS} \Leftrightarrow 2\sum_{cyc} x^4 \overset{?}{\geq} \sum_{cyc} x^3y + \sum_{cyc} xy^3 \to \text{true via}$$

$$\text{Schur} + \text{A} - \text{G} \therefore 3xyz\sum_{cyc} x^3 - \sum_{cyc} x^3y - \sum_{cyc} xy^3 + xyz\sum_{cyc} x - 2\sum_{cyc} xy \geq 0 \to \text{(i)}$$

$$\text{and again, } 3\sum_{cyc} x^3y^3 + \left(\sum_{cyc} x^3y + \sum_{cyc} xy^3 - 2\sum_{cyc} x^2\right) - \sum_{cyc} x^2y^2 - \sum_{cyc} xy - 3$$

$$\overset{\text{AM-GM \& via } (\bullet\bullet\bullet)}{\geq} 3\sum_{cyc} x^3y^3 - \sum_{cyc} x^2y^2 - \sum_{cyc} xy - 3 \overset{\text{Chebyshev and AM-GM}}{\geq}$$

$$\left(\frac{1}{3}.3\sqrt[3]{x^2y^2z^2}.\sum_{cyc} x^2y^2 - \sum_{cyc} x^2y^2\right) + \left(\frac{1}{3}.3\sqrt[3]{x^4y^4z^4}.\sum_{cyc} xy - \sum_{cyc} xy\right) +$$

$$3x^2y^2z^2 - 3 \overset{\text{via ②}}{\geq} 0 \therefore 3\sum_{cyc} x^3y^3 + \sum_{cyc} x^3y + \sum_{cyc} xy^3 - \sum_{cyc} x^2y^2 - 2\sum_{cyc} x^2$$

$$-\sum_{cyc} xy - 3 \to \text{(ii) and } 3x^2y^2z^2 - 3 + \sum_{cyc} x^2y^2 - \sum_{cyc} x^2 \overset{\text{via ② and } (\bullet\bullet\bullet)}{\geq} 0 \to \text{(iii)}$$

$$\therefore (\bullet) \text{ and (i)} + \text{(ii)} + \text{(iii)} \Rightarrow (*) \text{ is true } (\because \lambda \geq 0) \therefore \sum_{cyc} \frac{1}{a + \lambda.\sqrt{bc} + 1} \leq \frac{3}{\lambda + 2}$$

$$\forall\, a, b, c > 0 \,\Big|\, \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 3 \text{ and } \lambda \geq 0, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$