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If $a,b,c>0$, $a+b+c+2=abc$ and $\lambda\ge 1$, then :

$$\sum_{cyc}\frac{1}{\sqrt{a+\lambda}}\le\frac{3}{\sqrt{\lambda+2}}$$

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$$\sum_{cyc}\frac{1}{\sqrt{a+\lambda}}\overset{CBS}{\le}\sqrt{3\sum_{cyc}\frac{1}{a+\lambda}}=\sqrt{3.\frac{\sum_{cyc}ab+2\lambda\sum_{cyc}a+3\lambda^2}{\lambda^3+abc+\lambda^2\sum_{cyc}a+\lambda\sum_{cyc}ab}}$$

$$\overset{?}{\le}\frac{3}{\sqrt{\lambda+2}}\overset{a+b+c+2=abc}{\Leftrightarrow}3\left(\sum_{cyc}a+2\right)+3\lambda^3+3\lambda^2.\sum_{cyc}a+3\lambda.\sum_{cyc}ab\overset{?}{\ge}$$

$$(\lambda+2).\sum_{cyc}ab+(2\lambda^2+4\lambda).\sum_{cyc}a+6\lambda^2$$

$$\Leftrightarrow(\lambda^2-4\lambda+3)\left(\sum_{cyc}a\right)+2(\lambda-1)\left(\sum_{cyc}ab\right)\boxed{\overset{?}{\ge}\ ①}6(\lambda^2-1)$$

$$\text{Let }\sum_{cyc}a=m\text{ and }\because\lambda-1\ge0\therefore\text{LHS of ①}\ge$$

$$(\lambda^2-4\lambda+3)m+2(\lambda-1).\sqrt{3abc\sum_{cyc}a}$$

$$\overset{a+b+c+2=abc}{=}(\lambda^2-4\lambda+3)m+2(\lambda-1).\sqrt{3m(m+2)}\overset{?}{\ge}6(\lambda^2-1)$$

$$\Leftrightarrow(m-6)\lambda^2-\left(4m-2.\sqrt{3m(m+2)}\right)\lambda+3m+6-2.\sqrt{3m(m+2)}\boxed{\overset{?}{\ge}\ ②}0$$

$$\text{Now, }abc=\sum_{cyc}a+2\overset{AM-GM}{\ge}3.\sqrt[3]{abc}+2\Rightarrow t^3-3t-2\ge0\left(t=\sqrt[3]{abc}\right)\Rightarrow$$

$$(t-2)(t+1)^2\ge0\Rightarrow\sqrt[3]{abc}\ge2\Rightarrow\sum_{cyc}a+2\ge8\Rightarrow m\ge6,''='' \text{ iff } a=b=c=2$$

and so, when $m=6$, LHS of ② $=2.\sqrt{3m(m+2)}.(\lambda-1)\overset{\lambda\ge1}{\ge}0\Rightarrow$ ② is true for $m=6$ and we now consider $m>6$ and we have discriminant δ of LHS of ② $=$

$$\left(4m-2.\sqrt{3m(m+2)}\right)^2-4(m-6)\left(3m+6-2.\sqrt{3m(m+2)}\right)$$

$$=4\left(4m^2+18m+36-2(m+6).\sqrt{3m(m+2)}\right)$$

$$=\frac{8((2m^2+9m+18)^2-3m(m+2)(m+6)^2)}{2m^2+9m+18+(m+6).\sqrt{3m(m+2)}}$$

$$= \frac{8(m-6)^2(m+3)^2}{2m^2+9m+18+(m+6).\sqrt{3m(m+2)}} \overset{m>6}{>} 0$$

$$\therefore \sqrt{\delta} \boxed{\boxed{\overset{(*)}{=}}} 2.\sqrt{4m^2+18m+36-2(m+6).\sqrt{3m(m+2)}}$$ and we notice that, in order to prove ②, it suffices to prove : $\lambda \overset{?}{\geq} \frac{4m-2.\sqrt{3m(m+2)}+\sqrt{\delta}}{2(m-6)} \overset{\text{via }(*)}{\Leftrightarrow}$

$$\lambda(m-6)-2m+\sqrt{3m(m+2)} \boxed{\overset{?}{\underset{③}{\geq}}} \sqrt{4m^2+18m+36-2(m+6).\sqrt{3m(m+2)}}$$

Now, $\because m-6>0$ and $\lambda \geq 1 \therefore$ LHS of ③ $\geq m-6-2m+\sqrt{3m(m+2)}$

$$= \sqrt{3m(m+2)}-(m+6) = \sqrt{\left(\sqrt{3m(m+2)}-(m+6)\right)^2}$$

$$\left(\because \sqrt{3m(m+2)}-(m+6) = \frac{2(m-6)(m+3)}{\sqrt{3m(m+2)}+m+6} \overset{m>6}{>} 0\right)$$

$$= \sqrt{3m(m+2)+(m+6)^2-2(m+6).\sqrt{3m(m+2)}}$$

$$= \sqrt{4m^2+18m+36-2(m+6).\sqrt{3m(m+2)}} \Rightarrow ③ \text{ is true}$$

$\Rightarrow$ ② is true $\forall\, m \geq 6 \Rightarrow$ ① is true $\therefore \sum_{cyc} \frac{1}{\sqrt{a+\lambda}} \leq \frac{3}{\sqrt{\lambda+2}}$

$\forall\, a,b,c>0 \mid a+b+c+2=abc \wedge \lambda \geq 1$, $'' = ''$ iff $a=b=c=2$ (QED)