ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $xy + yz + zx + 2xyz = 1$, $\lambda \geq \frac{1}{2}$ then:

$$\sum \sqrt{\lambda - x^2} \leq \frac{3\sqrt{4\lambda - 1}}{2}$$

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$$xy + yz + zx + 2xyz = 1$$
$$xy + yz + zx + 2\left(\sqrt[3]{xyz}\right)^3 = 1$$
$$\frac{(x+y+z)^2}{3} + 2\left(\frac{(x+y+z)}{3}\right)^3 \overset{AM-GM}{\geq} 1$$
$$\frac{p^2}{3} + \frac{2p^3}{27} \overset{p=x+y+z}{\geq} 1 \; or \; 2p^3 + 9p^2 - 27 \geq 0$$
$$(2p-3)(p+3)^2 \geq 0 \; or \; 2p - 3 \geq 0 \; (since \; (p+3)^2 > 0)$$
$$or \; p \geq \frac{3}{2} \; or \; x + y + z \geq \frac{3}{2} \; (1)$$
$$\sum \sqrt{\lambda - x^2} \overset{CBS}{\leq} \sqrt{3\left(3\lambda - (x^2+y^2+z^2)\right)} \overset{CBS}{\leq} \sqrt{3\left(3\lambda - \frac{(x+y+z)^2}{3}\right)} \overset{(1)}{\leq}$$
$$\leq \sqrt{3\left(3\lambda - \frac{\left(\frac{3}{2}\right)^2}{3}\right)} = \sqrt{3\left(3\lambda - \frac{1}{3}\cdot\frac{9}{4}\right)} = \sqrt{\frac{9}{4}(4\lambda - 1)} = \frac{3\sqrt{4\lambda - 1}}{2}$$
$$Equality \; holds \; for \; x = y = z = \frac{1}{2}.$$