ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, xyz = 1$ and $\lambda \geq 0$ then :

$$\sum_{cyc} \frac{x^2}{x + \lambda y + y^3 z} \geq \frac{3}{\lambda + 2}$$

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$$\sum_{cyc} \frac{x^2}{x + \lambda y + y^3 z} = \sum_{cyc} \frac{x\left(x + \lambda y + y^3 z - (\lambda y + y^3 z)\right)}{x + \lambda y + y^3 z} =$$

$$\sum_{cyc} x - \sum_{cyc} \frac{xy(\lambda + y^2 z)}{\lambda y + x + y^3 z} \overset{\text{AM-GM}}{\geq} \sum_{cyc} x - \sum_{cyc} \frac{xy(\lambda + y^2 z)}{\lambda y + 2.\sqrt{xy^3 z}} \overset{xyz=1}{=}$$

$$\sum_{cyc} x - \frac{xy(\lambda + y^2 z)}{\lambda y + 2y} = \sum_{cyc} x - \frac{1}{\lambda + 2}.\left(\lambda \sum_{cyc} x + xyz \sum_{cyc} x\right)$$

$$\overset{xyz=1}{=} \sum_{cyc} x - \frac{1}{\lambda + 2}.\left(\lambda \sum_{cyc} x + \sum_{cyc} x\right) = \frac{\lambda \sum_{cyc} x + 2\sum_{cyc} x - \lambda \sum_{cyc} x - \sum_{cyc} x}{\lambda + 2}$$

$$= \frac{\sum_{cyc} x}{\lambda + 2} \overset{\text{AM-GM}}{\geq} \frac{3.\sqrt[3]{xyz}}{\lambda + 2} \overset{xyz=1}{=} \frac{3}{\lambda + 2} \therefore \sum_{cyc} \frac{x^2}{x + \lambda y + y^3 z} \geq \frac{3}{\lambda + 2}$$

$$\forall\, x, y, z > 0 \,\middle|\, xyz = 1 \text{ and } \lambda \geq 0, '' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$