ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ and $a + b = 2, \lambda \geq \frac{1}{2}$ then:

$$ab + \frac{1}{a^2 + b^2 + \lambda} - \frac{1}{a^3 + b^3 + \lambda} + \frac{1}{a^4 + b^4 + \lambda} \leq \frac{\lambda + 3}{\lambda + 2}$$

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$$ab \overset{AM-GM}{\leq} \left(\frac{a+b}{2}\right)^2 \overset{a+b=2}{=} 1 \ (1)$$

$$a^2 + b^2 = \frac{a^2}{1} + \frac{b^2}{1} \overset{Radon}{\geq} \frac{(a+b)^2}{2} \overset{a+b=2}{=} 2 \ \ (2)$$

$$\frac{a^4}{1^3} + \frac{b^4}{1^3} \overset{Radon}{\geq} \frac{(a+b)^4}{8} \overset{a+b=2}{=} 2 \ (3)$$

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b) \overset{a+b=2}{=} 8 - 6ab \ (4)$$

$$We\ need\ to\ show:$$

$$ab + \frac{1}{a^2 + b^2 + \lambda} - \frac{1}{a^3 + b^3 + \lambda} + \frac{1}{a^4 + b^4 + \lambda}$$

$$\overset{(2),(3)\&(4)}{\leq} \quad ab + \frac{1}{2 + \lambda} - \frac{1}{8 + \lambda - 6ab} + \frac{1}{\lambda + 2} = \frac{8ab + \lambda ab - 6ab - 1}{8 + \lambda - 6ab} + \frac{2}{\lambda + 2} =$$

$$= \frac{2ab + \lambda ab - 1}{8 + \lambda - 6ab} + \frac{2}{\lambda + 2} \overset{(1)}{\leq} \frac{2 + \lambda - 1}{8 + \lambda - 6} + \frac{2}{\lambda + 2} =$$

$$= \frac{1 + \lambda}{2 + \lambda} + \frac{2}{\lambda + 2} = \frac{\lambda + 3}{\lambda + 2}$$

Equality holds for a=b=1.