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If $a,b,c>0, abc=1$ and $\lambda\ge 2$ then :

$$\sum_{cyc}\frac{1}{\sqrt{a+\lambda^2-1}}\le\frac{3}{\lambda}$$

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$$\sum_{cyc}\frac{1}{\sqrt{a+\lambda^2-1}}\le\frac{3}{\lambda}\Leftrightarrow\sum_{cyc}\frac{1}{\sqrt{a+3+\lambda^2-4}}\le\frac{3}{\sqrt{4+\lambda^2-4}}$$

$$\Leftrightarrow\sum_{cyc}\frac{1}{\sqrt{a+3+m}}\overset{①}{\le}\frac{3}{\sqrt{4+m}}\ (\because m=\lambda^2-4)$$

$$\text{Now, }\sum_{cyc}\frac{1}{\sqrt{a+3+m}}\overset{CBS}{\le}\sqrt{3\sum_{cyc}\frac{1}{a+3+m}}\overset{?}{\le}\frac{3}{\sqrt{4+m}}\Leftrightarrow\sum_{cyc}\frac{1}{a+3+m}\overset{?}{\le}\frac{3}{4+m}$$

$$\Leftrightarrow 3\prod_{cyc}(a+3+m)\overset{?}{\ge}(m+4)\sum_{cyc}\left((b+3+m)(c+3+m)\right)$$

$$\Leftrightarrow m^2\left(\sum_{cyc}a-3\right)+m\left(2\sum_{cyc}ab+4\sum_{cyc}a-18\right)+$$

$$3abc+5\sum_{cyc}ab+3\sum_{cyc}a-27\overset{?}{\ge}0\to\text{true}\because m=\lambda^2-4\overset{\lambda\ge 2}{\ge}0\text{ and}$$

$$\because\sum_{cyc}a\overset{AM-GM}{\ge}3.\sqrt[3]{abc}\overset{abc=1}{\ge}3\text{ and also},\sum_{cyc}ab\overset{AM-GM}{\ge}3.\sqrt[3]{a^2b^2c^2}\overset{abc=1}{\ge}3$$

$$\Rightarrow\left(\sum_{cyc}a-3\right),\left(2\sum_{cyc}ab+4\sum_{cyc}a-18\right),\left(3abc+5\sum_{cyc}ab+3\sum_{cyc}a-27\right)\ge 0$$

$$\Rightarrow ①\text{ is true}\therefore\sum_{cyc}\frac{1}{\sqrt{a+\lambda^2-1}}\le\frac{3}{\lambda}\ \forall\ abc=1\text{ and }\lambda\ge 2,$$

$$''=''\text{ iff }a=b=c=1\ (QED)$$