ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, abc = 1$ and $\lambda \geq 2$ then :

$$\sum_{cyc} \frac{1}{\sqrt{a^3 + \lambda^2 - 1}} \leq \frac{3}{\lambda}$$

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$$a^3 = x, b^3 = y, c^3 = z\ (xyz = 1) \Rightarrow \sum_{cyc} \frac{1}{\sqrt{a^3 + \lambda^2 - 1}} \leq \frac{3}{\lambda}$$

$$\Leftrightarrow \sum_{cyc} \frac{1}{\sqrt{x + \lambda^2 - 1}} \leq \frac{3}{\lambda} \Leftrightarrow \sum_{cyc} \frac{1}{\sqrt{x + 3 + \lambda^2 - 4}} \leq \frac{3}{\sqrt{4 + \lambda^2 - 4}}$$

$$\Leftrightarrow \sum_{cyc} \frac{1}{\sqrt{x + 3 + m}} \overset{①}{\leq} \frac{3}{\sqrt{4 + m}}\ (\because m = \lambda^2 - 4)$$

$$\text{Now, } \sum_{cyc} \frac{1}{\sqrt{x + 3 + m}} \overset{CBS}{\leq} \sqrt{3 \sum_{cyc} \frac{1}{x + 3 + m}} \overset{?}{\leq} \frac{3}{\sqrt{4 + m}} \Leftrightarrow \sum_{cyc} \frac{1}{x + 3 + m} \overset{?}{\leq} \frac{3}{4 + m}$$

$$\Leftrightarrow 3 \prod_{cyc} (x + 3 + m) \overset{?}{\geq} (m + 4) \sum_{cyc} \big((y + 3 + m)(z + 3 + m)\big)$$

$$\Leftrightarrow m^2 \left(\sum_{cyc} x - 3 \right) + m \left(2 \sum_{cyc} xy + 4 \sum_{cyc} x - 18 \right) +$$

$$3xyz + 5 \sum_{cyc} xy + 3 \sum_{cyc} x - 27 \overset{?}{\geq} 0 \to \text{true} \because m = \lambda^2 - 4 \overset{\lambda \geq 2}{\geq} 0 \text{ and}$$

$$\because \sum_{cyc} x \overset{AM-GM}{\geq} 3.\sqrt[3]{xyz} \overset{xyz=1}{\geq} 3 \text{ and also}, \sum_{cyc} xy \overset{AM-GM}{\geq} 3.\sqrt[3]{x^2y^2z^2} \overset{xyz=1}{\geq} 3$$

$$\Rightarrow \left(\sum_{cyc} x - 3 \right), \left(2 \sum_{cyc} xy + 4 \sum_{cyc} x - 18 \right), \left(3xyz + 5 \sum_{cyc} xy + 3 \sum_{cyc} x - 27 \right) \geq 0$$

$$\Rightarrow ① \text{ is true} \therefore \sum_{cyc} \frac{1}{\sqrt{a^3 + \lambda^2 - 1}} \leq \frac{3}{\lambda}\ \forall\ abc = 1 \text{ and } \lambda \geq 2,$$

$$'' ='' \text{ iff } a = b = c = 1\ (\text{QED})$$