

ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z, t > 0$, $x + y + z + t = 4$ then:

$$\prod \left(1 + \frac{1}{x}\right)^{x^2} \geq 16$$

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Solution by Tapas Das-India

$$\begin{aligned} \log \left(1 + \frac{1}{x}\right)^{x^2} &= x^2 \log \left(1 + \frac{1}{x}\right) = x^2 \int_0^1 \frac{du}{x+u} \\ \log \prod_{cyc} \left(1 + \frac{1}{x}\right)^{x^2} &= \sum \log \left(1 + \frac{1}{x}\right)^{x^2} = \sum \frac{x^2}{x+u} du \stackrel{CBS}{\geq} \\ &\geq \int_0^1 \frac{(x+y+z+t)^2}{(x+y+z+t)+4u} du \stackrel{x+y+z+t=4}{=} \int_0^1 \frac{4^2}{4+4u} du = \\ &= 4 \int_0^1 \frac{du}{u+1} = 4(\log|u+1|)_0^1 = 4 \log 2 = \log 16 \\ \prod \left(1 + \frac{1}{x}\right)^{x^2} &\geq 16 \end{aligned}$$

Equality holds for $x=y=z=t=1$.