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If $a,\mathrm{b},\mathrm{c}>0, a\mathrm{bc}=1$ and $\lambda\geq 0$ then :

$$\sum_{cyc}\sqrt{\lambda a^2+\mathrm{b}(1+a)}\geq 3\sqrt{\lambda+2}$$

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$$\left(\sum_{cyc}\sqrt{\lambda a^2+\mathrm{b}(1+a)}\right)^2=$$

$$\lambda\sum_{cyc}a^2+\sum_{cyc}a+\sum_{cyc}a\mathrm{b}+2.\sum_{cyc}\left(\sqrt{\lambda a^2+\mathrm{b}(1+a)}.\sqrt{\lambda\mathrm{b}^2+\mathrm{c}(1+\mathrm{b})}\right)$$

$$\overset{\substack{\text{Reverse CBS}\\ \text{and }\because\lambda\geq 0}}{\geq}\lambda\sum_{cyc}a^2+\sum_{cyc}a+\sum_{cyc}a\mathrm{b}+2.\sum_{cyc}\left(\lambda a\mathrm{b}+\sqrt{\mathrm{bc}(1+a)(1+\mathrm{b})}\right)$$

$$\overset{a\mathrm{bc}=1}{=}\lambda\left(\sum_{cyc}a^2+2\sum_{cyc}a\mathrm{b}\right)+\sum_{cyc}a+\sum_{cyc}a\mathrm{b}+2\sum_{cyc}\sqrt{(1+\mathrm{b})\left(\frac{1}{a}+1\right)}$$

$$\overset{\text{Reverse CBS}}{\geq}\lambda\left(\sum_{cyc}a\right)^2+\sum_{cyc}a+\sum_{cyc}a\mathrm{b}+2\sum_{cyc}\left(\frac{1}{\sqrt{a}}+\sqrt{\mathrm{b}}\right)\overset{\substack{\text{AM}-\text{GM}\\ \text{and }\because\lambda\geq 0}}{\geq}$$

$$\lambda\left(9.\sqrt[3]{a^2\mathrm{b}^2\mathrm{c}^2}\right)+3.\sqrt[3]{a\mathrm{bc}}+3.\sqrt[3]{a^2\mathrm{b}^2\mathrm{c}^2}+2\left(\frac{3}{\sqrt[3]{\sqrt{a\mathrm{bc}}}}+3.\sqrt[3]{\sqrt{a\mathrm{bc}}}\right)\overset{a\mathrm{bc}=1}{=}$$

$$9\lambda+3+3+2(3+3)=9(\lambda+2)\Rightarrow\sum_{cyc}\sqrt{\lambda a^2+\mathrm{b}(1+a)}\geq 3.\sqrt{\lambda+2}$$

$$\forall\, a,\mathrm{b},\mathrm{c}>0\,\Big|\; a\mathrm{bc}=1,''=''\text{ iff }a=\mathrm{b}=\mathrm{c}=1\ (\text{QED})$$