ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, $\lambda\geq 0$ then:

$$\sum_{cyc}\frac{b-a}{a+\lambda}\geq 0$$

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$$\sum_{cyc}\frac{b-a}{a+\lambda}\geq 0\rightarrow\frac{b-a}{a+\lambda}+\frac{c-b}{b+\lambda}+\frac{a-c}{c+\lambda}\geq 0\rightarrow$$

$$\frac{b}{a+\lambda}+\frac{c}{b+\lambda}+\frac{a}{c+\lambda}\geq\frac{a}{a+\lambda}+\frac{b}{b+\lambda}+\frac{c}{c+\lambda}\quad(*)$$

$$\textit{Prove that } (*) \textit{ is true}:$$

$$\frac{b(b+\lambda)(c+\lambda)+c(a+\lambda)(c+\lambda)+a(a+\lambda)(b+\lambda)}{(a+\lambda)(b+\lambda)(c+\lambda)}\geq$$

$$\frac{a(b+\lambda)(c+\lambda)+b(a+\lambda)(c+\lambda)+c(a+\lambda)(b+\lambda)}{(a+\lambda)(b+\lambda)(c+\lambda)}$$

$$b^2c+b^2\lambda+bc\lambda+b\lambda^2+ac^2+ac\lambda+c^2\lambda+c\lambda^2+a^2b+a^2\lambda+ba\lambda+a\lambda^2\geq$$

$$abc+ba\lambda+ac\lambda+a\lambda^2+abc+ba\lambda+bc\lambda+b\lambda^2+abc+ac\lambda+bc\lambda+c\lambda^2$$

$$a^2b+b^2c+ac^2+(a^2+b^2+c^2)\lambda\geq 3abc+(ab+bc+ac)\lambda$$

$$(a^2b+b^2c+ac^2-3abc)+((a^2+b^2+c^2)-(ab+bc+ac))\lambda\geq 0$$

$$\textit{According to Muirhead's inequality}:$$

$$(2;1;0)>(1;1;1)\quad \textit{and}\quad a^2+b^2+c^2\geq ab+bc+ac$$

$$a^2b+b^2c+ac^2\geq 3abc\ \textit{ and }\ a^2+b^2+c^2\geq ab+bc+ac$$

$$\textit{Equality holds for}: a=b=c$$