ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x^2 + y^2 + z^2 = 3$ *and* $\lambda \geq 0$ then :

$$\sum_{cyc} \frac{x^5}{x^3 + x + \lambda} \geq \frac{3}{\lambda + 2}$$

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$$\frac{x^5}{x^3 + x + \lambda} \overset{?}{\geq} \frac{1}{\lambda + 2} + \frac{(x^2 - 1)(6 + 5\lambda)}{2(\lambda + 2)^2}$$

$$\Leftrightarrow \frac{\lambda x^5 + 2x^5 - x^3 - x - \lambda}{(\lambda + 2)(x^3 + x + \lambda)} \overset{?}{\geq} \frac{(x^2 - 1)(6 + 5\lambda)}{2(\lambda + 2)^2}$$

$$\Leftrightarrow \frac{\lambda(x - 1)(x^4 + x^3 + x^2 + x + 1) + x(x - 1)(x + 1)(2x^2 + 1)}{(\lambda + 2)(x^3 + x + \lambda)} \overset{?}{\geq} \frac{(x - 1)(x + 1)(6 + 5\lambda)}{2(\lambda + 2)^2}$$

$$\Leftrightarrow \frac{x - 1}{\lambda + 2} \cdot \left(\frac{\lambda(x^4 + x^3 + x^2 + x + 1) + x(x + 1)(2x^2 + 1)}{x^3 + x + \lambda} - \frac{(x + 1)(6 + 5\lambda)}{2(\lambda + 2)}\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{x - 1}{\lambda + 2} \cdot \frac{\lambda^2(2x^4 + 2x^3 + 2x^2 - 3x - 3) + \lambda(3x^4 + 3x^3 + x^2 - 5x - 2) + 2(x^4 + x^3 - x^2 - x)}{2(\lambda + 2)(x^3 + x + \lambda)} \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{x - 1}{\lambda + 2} \cdot \frac{\lambda^2(x - 1)(2x^3 + 4x^2 + 6x + 3) + \lambda(x - 1)(3x^3 + 6x^2 + 7x + 2) + 2x(x - 1)(x + 1)^2}{2(\lambda + 2)(x^3 + x + \lambda)} \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{(x - 1)^2 . (\lambda^2(2x^3 + 4x^2 + 6x + 3) + \lambda(3x^3 + 6x^2 + 7x + 2) + 2x(x + 1)^2)}{2(\lambda + 2)^2(x^3 + x + \lambda)} \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because x > 0 \text{ and } \lambda \geq 0 \therefore \frac{x^5}{x^3 + x + \lambda} \geq \frac{1}{\lambda + 2} + \frac{(x^2 - 1)(6 + 5\lambda)}{2(\lambda + 2)^2} \text{ and analogs}$$

$$\therefore \sum_{cyc} \frac{x^5}{x^3 + x + \lambda} \geq \frac{3}{\lambda + 2} + \sum_{cyc} \frac{(x^2 - 1)(6 + 5\lambda)}{2(\lambda + 2)^2}$$

$$= \frac{3}{\lambda + 2} + \frac{6 + 5\lambda}{2(\lambda + 2)^2} . \left(\sum_{cyc} x^2 - 3\right) \overset{x^2 + y^2 + z^2 = 3}{=} \frac{3}{\lambda + 2}$$

$$\therefore \sum_{cyc} \frac{x^5}{x^3 + x + \lambda} \geq \frac{3}{\lambda + 2} \ \forall\, x, y, z > 0 \,\middle|\, x^2 + y^2 + z^2 = 3 \text{ and } \lambda \geq 0,$$

$$'' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$