ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, \sum_{cyc} \frac{1}{a^3} = 3$ and $\lambda \in \mathbb{N}$ then :

$$\sum_{cyc} \sqrt{\frac{1}{\lambda a + (\lambda+1)b^2 + (\lambda+2)c^3}} \le \sqrt{\frac{3}{\lambda+1}}$$

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$$Via \text{ Power} - \text{Mean Inequality}, \left(\frac{\sum_{cyc} \frac{1}{a^3}}{3}\right)^{\frac{1}{3}} \ge \left(\frac{\sum_{cyc} \frac{1}{a^2}}{3}\right)^{\frac{1}{2}}$$

$$\Rightarrow \frac{\sum_{cyc} \frac{1}{a^2}}{3} \le \left(\frac{\sum_{cyc} \frac{1}{a^3}}{3}\right)^{\frac{2}{3}} \overset{\sum_{cyc} \frac{1}{a^3} = 3}{=} 1 \Rightarrow \sum_{cyc} \frac{1}{a^2} \le 3 \to (1) \text{ and via Holder},$$

$$\sum_{cyc} \frac{1}{a^3} \ge \frac{1}{9}\left(\sum_{cyc} \frac{1}{a}\right)^3 \overset{\sum_{cyc} \frac{1}{a^3} = 3}{\Rightarrow} 27 \ge \left(\sum_{cyc} \frac{1}{a}\right)^3 \Rightarrow \sum_{cyc} \frac{1}{a} \le 3 \to (2)$$

$$Via \text{ Weigted AM} - \text{Weighted HM inequality},$$

$$\frac{\lambda a + (\lambda+1)b^2 + (\lambda+2)c^3}{\lambda + \lambda + 1 + \lambda + 2} \ge \frac{\lambda + \lambda + 1 + \lambda + 2}{\frac{\lambda}{a} + \frac{\lambda+1}{b^2} + \frac{\lambda+2}{c^3}}$$

$$\Rightarrow \sqrt{\frac{1}{\lambda a + (\lambda+1)b^2 + (\lambda+2)c^3}} \le \frac{1}{3(\lambda+1)} \cdot \sqrt{\frac{\lambda}{a} + \frac{\lambda+1}{b^2} + \frac{\lambda+2}{c^3}} \text{ and analogs}$$

$$\therefore \sum_{cyc} \sqrt{\frac{1}{\lambda a + (\lambda+1)b^2 + (\lambda+2)c^3}} \le \frac{1}{3(\lambda+1)} \cdot \sum_{cyc} \sqrt{\frac{\lambda}{a} + \frac{\lambda+1}{b^2} + \frac{\lambda+2}{c^3}} \overset{CBS}{\le}$$

$$\frac{\sqrt{3}}{3(\lambda+1)} \cdot \sqrt{\lambda \sum_{cyc} \frac{1}{a} + (\lambda+1) \sum_{cyc} \frac{1}{a^2} + (\lambda+2) \sum_{cyc} \frac{1}{a^3}} \overset{\text{via (1) and (2)}}{\le}$$

$$\frac{\sqrt{3}}{3(\lambda+1)} \cdot \sqrt{3\lambda + 3(\lambda+1) + 3(\lambda+2)} = \sqrt{\frac{3}{\lambda+1}}$$

$$\therefore \sum_{cyc} \sqrt{\frac{1}{\lambda a + (\lambda+1)b^2 + (\lambda+2)c^3}} \le \sqrt{\frac{3}{\lambda+1}} \ \forall a, b, c > 0 \mid \sum_{cyc} \frac{1}{a^3} = 3 \text{ and } \lambda \in \mathbb{N},$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$