ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, a^2 + b^2 + c^2 = 3$ *and* $\lambda \geq 0$ then :

$$\sum_{cyc} \frac{a}{a^2 + 2bc + \lambda a} \leq \frac{3}{\lambda + 3abc}$$

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$$\forall\, m > 0, \frac{a}{m + \lambda a} \overset{?}{\leq} \frac{1}{m + \lambda} + (a-1).\frac{m}{(m+\lambda)^2}$$

$$\Leftrightarrow \frac{m(a-1)}{(m+\lambda)^2} \overset{?}{\geq} \frac{am + a\lambda - m - a\lambda}{(m+\lambda)(m+\lambda a)} = \frac{m(a-1)}{(m+\lambda)(m+\lambda a)}$$

$$\Leftrightarrow \frac{m(a-1)}{m+\lambda}.\left(\frac{1}{m+\lambda} - \frac{1}{m+\lambda a}\right) \overset{?}{\geq} 0 \Leftrightarrow \frac{m\lambda(a-1)^2}{(m+\lambda)^2(m+\lambda a)} \overset{?}{\geq} 0 \rightarrow \text{true} \because m > 0$$

$$\textit{and } \lambda \geq 0 \therefore \frac{a}{m+\lambda a} \leq \frac{1}{m+\lambda} + (a-1).\frac{m}{(m+\lambda)^2}\ \forall\, m > 0$$

$$\textit{and via } m \equiv 3.\sqrt[3]{a^2b^2c^2}, \text{we get :}$$

$$\frac{a}{3.\sqrt[3]{a^2b^2c^2} + \lambda a} \leq \frac{1}{3.\sqrt[3]{a^2b^2c^2} + \lambda} + (a-1).\frac{3.\sqrt[3]{a^2b^2c^2}}{\left(3.\sqrt[3]{a^2b^2c^2} + \lambda\right)^2}\ \&\ \textit{analogs} \rightarrow (1)$$

$$\text{Now}, \sum_{cyc} \frac{a}{a^2 + 2bc + \lambda a} \overset{\text{AM-GM}}{\leq} \sum_{cyc} \frac{a}{3.\sqrt[3]{a^2b^2c^2} + \lambda a} \overset{\text{via (1)}}{\leq}$$

$$\frac{3}{3.\sqrt[3]{a^2b^2c^2} + \lambda} + \left(\sum_{cyc} a - 3\right).\frac{3.\sqrt[3]{a^2b^2c^2}}{\left(3.\sqrt[3]{a^2b^2c^2} + \lambda\right)^2} \overset{\text{CBS}}{\leq}$$

$$\frac{3}{3.\sqrt[3]{a^2b^2c^2} + \lambda} + \left(\sqrt{3\sum_{cyc} a^2} - 3\right).\frac{3.\sqrt[3]{a^2b^2c^2}}{\left(3.\sqrt[3]{a^2b^2c^2} + \lambda\right)^2} \overset{a^2+b^2+c^2=3}{=}$$

$$\frac{3}{3.\sqrt[3]{a^2b^2c^2} + \lambda} \overset{?}{\leq} \frac{3}{\lambda + 3abc} \Leftrightarrow abc \overset{?}{\leq} \sqrt[3]{a^2b^2c^2} \Leftrightarrow abc \overset{?}{\leq} 1 \rightarrow \text{true}$$

$$\because 3 = \sum_{cyc} a^2 \overset{\text{AM-GM}}{\geq} 3.\sqrt[3]{a^2b^2c^2} \Rightarrow abc \leq 1$$

$$\therefore \sum_{cyc} \frac{a}{a^2 + 2bc + \lambda a} \leq \frac{3}{\lambda + 3abc}\ \forall\, a, b, c > 0 \,\big|\, a^2 + b^2 + c^2 = 3 \textit{ and } \lambda \geq 0,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$