# ROMANIAN MATHEMATICAL MAGAZINE

**If $x, y, z > 0, x + y + z = 3$ *and* $\lambda \geq 1$ then :**

$$\sum_{cyc} \frac{x}{x^2 + x + \lambda} \leq \frac{3}{\lambda + 1 + xyz}$$

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$$\frac{x}{x^2 + x + \lambda} \overset{?}{\leq} \frac{1}{\lambda + 2} + \frac{(x-1)(\lambda-1)}{(\lambda+2)^2}$$

$$\Leftrightarrow \frac{x\lambda + 2x - x^2 - x - \lambda}{(\lambda+2)(x^2 + x + \lambda)} \overset{?}{\leq} \frac{(x-1)(\lambda-1)}{(\lambda+2)^2}$$

$$\Leftrightarrow \frac{(\lambda - x)(x-1)}{(\lambda+2)(x^2+x+\lambda)} \overset{?}{\leq} \frac{(x-1)(\lambda-1)}{(\lambda+2)^2} \Leftrightarrow \frac{x-1}{\lambda+2} \cdot \left(\frac{\lambda - x}{x^2 + x + \lambda} - \frac{\lambda - 1}{\lambda + 2}\right) \overset{?}{\leq} 0$$

$$\Leftrightarrow \frac{x-1}{\lambda+2} \cdot \left(\frac{x^2 - x - \lambda(x^2 + 2x - 3)}{(\lambda+2)(x^2+x+\lambda)}\right) \overset{?}{\leq} 0$$

$$\Leftrightarrow \frac{x-1}{\lambda+2} \cdot \left(\frac{x(x-1) - \lambda(x-1)(x+3)}{(\lambda+2)(x^2+x+\lambda)}\right) \overset{?}{\leq} 0$$

$$\Leftrightarrow \frac{-(x-1)^2}{\lambda+2} \cdot \frac{x(\lambda-1) + 3\lambda}{(\lambda+2)(x^2+x+\lambda)} \overset{?}{\leq} 0 \rightarrow \text{true} \because \lambda \geq 1 \text{ and } x > 0$$

$$\therefore \frac{x}{x^2+x+\lambda} \leq \frac{1}{\lambda+2} + \frac{(x-1)(\lambda-1)}{(\lambda+2)^2} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} \frac{x}{x^2+x+\lambda} \leq \frac{3}{\lambda+2} + \frac{(\lambda-1)}{(\lambda+2)^2} \cdot \left(\sum_{cyc} x - 3\right)$$

$$\overset{x+y+z=3}{=} \frac{3}{\lambda+2} \overset{?}{\leq} \frac{3}{\lambda+1+xyz} \Leftrightarrow xyz \overset{?}{\leq} 1 \rightarrow \text{true} \because 3 = \sum_{cyc} x \overset{\text{AM-GM}}{\geq} 3.\sqrt[3]{xyz}$$

$$\Rightarrow xyz \leq 1 \therefore \sum_{cyc} \frac{x}{x^2+x+\lambda} \leq \frac{3}{\lambda+1+xyz} \; \forall \, x, y, z > 0 \, \middle| \, x + y + z = 3 \text{ and } \lambda \geq 1,$$

$$'' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$